Port-Hamiltonian Dynamics on Graphs: Consensus and Coordination Control

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Abstract

The purpose of this paper is to provide a new angle to some existing results on consensus algorithms and coordination control strategies, by showing how the resulting dynamical systems can be described within a port-Hamiltonian framework. In particular we will show how on each directed graph there is a canonically defined Dirac structure relating the variables in the vertex and edge space. This Dirac structure, called the vertex-edge Dirac structure, is underlying the subsequent port-Hamiltonian description of a standard consensus algorithm (with leader and follower agents), as well as the port-Hamiltonian formulation of the coordination control scheme (based on passivity) given in the work of Arcak. We will show how the port-Hamiltonian formulation unifies existing results, and how it may lead to new insights and generalizations.

1. Introduction

Let $\mathcal{G}$ be a directed graph, 'graph' in the sequel, specified by its *incidence matrix* B.

- Define its *vertex space* Λ_0 as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the sequel, $\mathcal{R}$ will be either $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = \mathbb{R}^3$. In the first case, Λ_0 can be identified with $\mathbb{R}^{\bar{v}}$.
- Define its *edge space* Λ_1 as the vector space of all functions from $\mathcal{E}$ to the same space $\mathcal{R}$. Again, if $\mathcal{R} = \mathbb{R}$ then Λ_1 can be identified with $\mathbb{R}^{\bar{e}}$.
- The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0, respectively Λ^1.
- The incidence matrix B induces the *incidence operator* $B_{\mathcal{R}} : \Lambda_1 \to \Lambda_0$, defined by the matrix
$$B_{\mathcal{R}} := B \otimes E$$
where $E : \mathcal{R} \to \mathcal{R}$ is the identity matrix and $\otimes$ denotes the Kronecker product. For $\mathcal{R} = \mathbb{R}$ the incidence operator reduces to the linear map given by the matrix B itself.
- The adjoint map $B^*_{\mathcal{R}} : \Lambda^0 \to \Lambda^1$ is called the *co-incidence* operator; for $\mathcal{R} = \mathbb{R}$ given by the transposed matrix B^T.

2. Open graphs and the vertex-edge Dirac structure

An *open graph* $\mathcal{G}$ is obtained from an ordinary graph by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of *boundary vertices*. These are the vertices that are open to interconnection. The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the *internal vertices* of the open graph.

The splitting of the vertices into internal and boundary vertices induces a splitting of the vertex space and its dual:
$$\begin{aligned}\Lambda_0 &= \Lambda_i \oplus \Lambda_b \\ \Lambda^0 &= \Lambda^i \oplus \Lambda^b\end{aligned}$$
while the incidence operator $B_{\mathcal{R}} : \Lambda_1 \to \Lambda_0$ splits as
$$B_{\mathcal{R}} = B_{\mathcal{R}i} \oplus B_{\mathcal{R}b}$$
with $B_{\mathcal{R}i} : \Lambda_1 \to \Lambda_i$ and $B_{\mathcal{R}|} : \Lambda_1 \to \Lambda_b$.

There is a canonically defined Dirac structure on the vertex and edge space of any directed graph. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ for some vector space $\mathcal{F}$ is a *Dirac structure* if
$$\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$$
where ${}^{\perp\!\!\perp}$ is the orthogonal complement with respect to the indefinite inner product $\ll \cdot, \cdot \gg$ on $\mathcal{F} \times \mathcal{F}^*$ defined as
$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1 \mid f_2 > + < e_2 \mid f_1 >,$$
with $f_1, f_2 \in \mathcal{F}, e_1, e_2 \in \mathcal{F}^*$, where $< \cdot \mid \cdot >$ denotes the duality product. In the finite-dimensional case an equivalent characterization is:
$$\begin{aligned}&(i) < f^* \mid f >= 0, \quad \text{for all } (f, f^*) \in \mathcal{D} \\ &(ii) \dim \mathcal{D} = \dim \mathcal{F}\end{aligned}$$

Any open graph $\mathcal{G}$ admits the *vertex-edge Dirac structure*
$$\begin{aligned}\mathcal{D}_{ve}(\mathcal{G}) = \{&(f, e, f_i, e_i, f_b, e_b) \in \\ &\Lambda_1 \times \Lambda^1 \times \Lambda_i \times \Lambda^i \times \Lambda_b \times \Lambda^b \mid \\ &B_{\mathcal{R}i} f = -f_i, B_{\mathcal{R}b} f = -f_b, e = B^*_{\mathcal{R}i} e_i + B^*_{\mathcal{R}b} e_b\}\end{aligned}$$

3. Consensus algorithms as port-Hamiltonian systems

Consider a network of agents moving in linear space $\mathcal{R}$, whose interaction topology is described by an *undirected* graph $\mathcal{G}$ (symmetric interaction). Distinguish between *leader* and *follower* agents, and associate the leader agents to the boundary and the follower agents to the internal vertices.

Associated to each agent v there is a vector $x_v \in \mathcal{R}$ describing the motion in the linear space $\mathcal{R}$. In a standard set-up, x_v satisfies the following dynamics
$$\dot{x}_v(t) = -\sum_{(v,w) \in E(\mathcal{G})} g_{(v,w)}(x_v(t) - x_w(t))$$
where $g_{(v,w)} > 0$ denotes a certain positive-definite *weight* matrix associated to each edge.

For simplicity of notation take $\mathcal{R} = \mathbb{R}$. Collecting all *follower* variables x_v into one vector $x \in \mathbb{R}^{\bar{v}_i}$, and all *leader* variables x_v into one vector $u \in \mathbb{R}^{\bar{v}_b}$ overall dynamics becomes
$$\dot{x} = -B_i G B_i^T x - B_i G B_b^T u$$
with G the diagonal matrix with elements $g_{(v,w)}$ for each edge (v, w).

This defines a port-Hamiltonian system with respect to the vertex-edge Dirac structure $\mathcal{D}_{ve}(\mathcal{G})$ and the Hamiltonian
$$H(x) = \frac{1}{2} \parallel x \parallel^2$$
leading to the energy storage equations
$$\dot{x} = -f_i, \quad e_i = \frac{\partial H}{\partial x}(x) = x$$
Furthermore, let the variables f, e satisfy the energy dissipation relations
$$f = -Ge$$
Then the consensus dynamics is given as the port-Hamiltonian system
$$\begin{aligned}\dot{x} &= -B_i G B_i^T \tfrac{\partial H}{\partial x}(x) - B_i G B_b^T u \\ y &= B_b G B_i^T \tfrac{\partial H}{\partial x}(x) + B_b G B_b^T u\end{aligned}$$
with $u = e_b \in \Lambda^b$, $y = f_b \in \Lambda_b$. The *output* y appears in the energy balance
$$\frac{d}{dt} H = -\begin{bmatrix} x^T & u^T \end{bmatrix} BGB^T \begin{bmatrix} x \\ u \end{bmatrix} + y^T u \leq y^T u,$$
and thus is seen to be a *passive output*.

Hence a physical analogue of the consensus dynamics is the dynamics of a number of unit masses, corresponding to each internal vertex, with linear dampers associated to the edges, and externally prescribed boundary velocities $u = e_b$, with outputs $y = f_b$ being the boundary forces.

In case of a *connected* graph there exists for each fixed vector $\bar{u}$ a unique equilibrium vector $\bar{x}$ such that
$$0 = -B_i G B_i^T \bar{x} - B_i G B_b^T \bar{u}$$
Asymptotic stability of $\bar{x}$ can be immediately proved by invoking the following *shifted passivity* property of the port-Hamiltonian system. Define the *shifted Hamiltonian* $V(x) := \frac{1}{2}(x - \bar{x})^T (x - \bar{x})$, then
$$\frac{d}{dt} V(x) = -\begin{bmatrix} (x - \bar{x})^T & (u - \bar{u})^T \end{bmatrix} BGB^T \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix} + (y - \bar{y})^T (u - \bar{u})$$
where $\bar{y} = B_b G B_i^T \bar{x} + B_b G B_b^T \bar{u}$ is the output equilibrium value. Hence for $u = \bar{u}$ the system converges to the maximal invariant set contained in
$$\{x \mid (x - \bar{x})^T B_i G B_i^T (x - \bar{x}) = 0\}$$
which for a connected graph is equal to the single point $\bar{x}$.

4. Port-Hamiltonian formulation of coordination control

Consider an open graph where each internal *vertex* corresponds to a port-Hamiltonian system. Coordination will be sought by designing a port-Hamiltonian dynamics associated to each *edge*. In the simplest case where the dynamics associated to each internal vertex v is given by
$$\begin{aligned}\dot{p}_v &= u_v \\ y_v &= \frac{\partial H_v}{\partial p_v}(p_v)\end{aligned}$$
These are coupled to the vertex-edge Dirac structure $\mathcal{D}_{ve}$ by setting
$$\begin{aligned}(u_1, \ldots, u_{\bar{v}}) &= -f_i \\ (y_1, \ldots, y_{\bar{v}}) &= e_i\end{aligned}$$
Now design the port-Hamiltonian dynamics associated to each edge e as
$$\begin{aligned}\dot{q}_e &= w_e \\ z_e &= \frac{\partial H_e}{\partial q_e}(q_e)\end{aligned}$$
for some Hamiltonian H_e, and couple the port-Hamiltonian vertex and edge integrator systems to the vertex-edge Dirac structure:
$$\begin{aligned}(w_1, \ldots, w_{\bar{e}}) &= e \\ (z_1, \ldots, z_{\bar{e}}) &= -f\end{aligned}$$
The resulting interconnected system is then given as the port-Hamiltonian system
$$\begin{aligned}\dot{q} &= B_i^T \frac{\partial H}{\partial p}(q, p) + B_b^T e_b \\ \dot{p} &= -B_i \frac{\partial H}{\partial q}(q, p) \\ f_b &= B_b^T \frac{\partial H}{\partial q}(q, p)\end{aligned}$$
where $q = (q_1, \ldots, q_{\bar{v}})$ and $p = (p_1, \ldots, p_{\bar{e}})$, and $H(q, p) := \sum H_v + \sum H_e$ denotes the total Hamiltonian. Coordination control is now sought to be achieved by designing the Hamiltonians H_e in such a way that the minimum of $H(q, p) := \sum H_v + \sum H_e$ corresponds to the desired formation.

In a typical formation control context, p_v is a *momentum* vector, q_e is a *configuration* vector, e_b an external (reference) *velocity* vector, and f_b the corresponding generalized *force* vector. Thus the total system is a *mass-spring* system with masses corresponding to the vertices and springs corresponding to the edges.

5. Conclusions

Standard (symmetric) consensus algorithms and coordination control strategies have been formulated as port-Hamiltonian dynamics with respect to a canonically defined Dirac structure associated with the incidence matrix. This leads to new interpretations and insights; in particular, the compositionality properties of consensus dynamics and coordination control, and the coordination control of unstable agents within this framework are currently under investigation.

Another venue for future research is the fact that *another* canonical Dirac structure on directed graphs, called the *Kirchhoff-Dirac structure*, may be derived from the vertex-edge Dirac structure, giving rise to other port-Hamiltonian dynamics. This Dirac structure naturally appears in the port-Hamiltonian description of the dynamics of RLC-circuits or in the structure-preserving discretization of infinite-dimensional physical phenomena. This provides additional input for considering consensus dynamics and coordination control in relation with physical dynamics.

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