A Nyquist criterion for synchronization in networks of heterogeneous linear systems

E. Lovisari lovisari@dei.unipd.it, U.T. Jönsson ulfj@math.kth.se

Problem setup

We are given a **network of n interacting agents**:

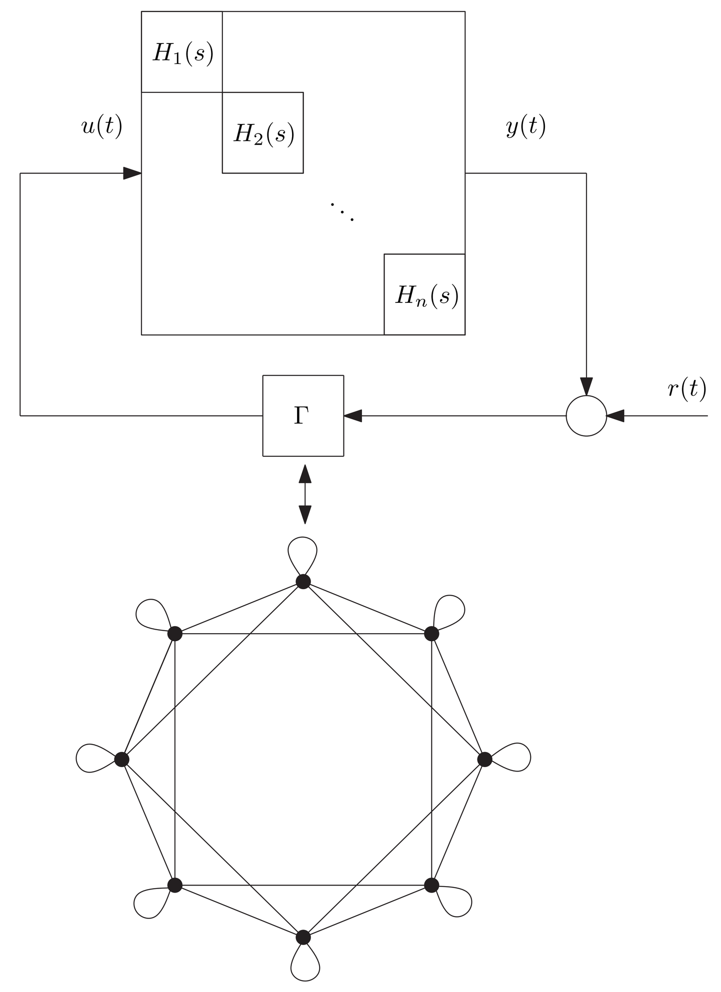
- agent $k \to$ SISO LTI system

$$y_k(t) = H_k(s)u_k(t),\ k = 1, \ldots, n$$

- Network $\iff$ graph $\mathcal{G} = (V, \mathcal{E})$, $V = \{1, \ldots, n\}$ and

$$(k, j) \in \mathcal{E} \iff j \text{ receives information from } k$$

- Interconnected system $\begin{cases} y = Hu \\ u = \Gamma y + r \end{cases}$
- $H = \mathrm{diag}(H_k(s) : k = 1, \ldots, n)$
- $\Gamma \in \mathbb{R}^{n\times n}$ normal, $\Gamma \mathbf{1} = 0$, $\dim\ker\Gamma = 1$



Goal

We want to give sufficient conditions on Γ and $H_k(s)$ in order to **synchronize** the network

$$\|y_i(t) - y_j(t)\| \overset{t\to\infty}{\longrightarrow} 0.$$

Remark: once the system is synchronized, the input signal r can be easily used in order to perform higher level tasks, e.g. formation control.

Prior work

Fax, J.A. and Murray, R.M., *Information flow and cooperative control of vehicle formations*, TAC 2004: if $H_k(s) = H(s)$, synchronization holds if **Nyquist criterion** holds for $-\frac{1}{\lambda_k}$, for any nonzero eigenvalue λ_k of Γ.

Main Result

Assume $H_k(s) = N_0(s) + N_k(s)$, where $N_0(s)$ is the "nominal" plant, $N_k(s)$ is a perturbation.

A **sufficient condition** for $e^{\alpha t}y(t) \overset{t\to\infty}{\longrightarrow} \text{span}\{\mathbf{1}\}$ for any input r which satisfies $e^{\alpha t}r(t), e^{\alpha t}\dot{r}(t) \in \mathbf{L}_2[0, \infty)$ is that α is chosen such that:

i) $W_0(s-\alpha) = \frac{N_0(s-\alpha)}{1-N_0(s-\alpha)\lambda_k}$ is stable $\forall\, \lambda_k \neq 0$ eigenvalue if Γ, $1 - N_0(s-\alpha)\lambda_k$ is nonsingular on the imaginary axis, $\frac{N_k(s-\alpha)}{N_0(s-\alpha)}$ are stable $\forall\, k$

ii) it holds $\mathcal{N}\left[H_0, \ldots, H_n\right](j\omega - \alpha) \cap \Omega = \emptyset, \forall\, \omega \in \mathbf{R} \cup \{\infty\}$

	Direct $3-D$	Inverse $3-D$	Direct $2-D$	Inverse $2-D$
$\mathcal{N}$	$\mathrm{co}\{(\mathrm{Re}H_k, \mathrm{Im}H_k, \lvert H_k\rvert^2), \forall\, k\}$	$\mathrm{co}\left\{\left(\mathrm{Re}\frac{1}{H_k}, \mathrm{Im}\frac{1}{H_k}, \frac{1}{\lvert H_k\rvert^2}\right), \forall\, k\right\}$	$\mathrm{co}\{H_1(j\omega), \ldots, H_n(j\omega)\}$	$\mathrm{co}\left\{\frac{1}{H_1(j\omega)}, \ldots, \frac{1}{H_n(j\omega)}\right\}$
Ω	$(0, 0, \mathbb{R}^+) + \mathrm{co}\left\{\left(\mathrm{Re}\frac{1}{\lambda_k}, \mathrm{Im}\frac{1}{\lambda_k}, \frac{1}{\lvert\lambda_k\rvert^2}\right), \forall\, k\right\}$	$\mathrm{co}\left\{(\mathrm{Re}\lambda_k, \mathrm{Im}\lambda_k, \lvert\lambda_k\rvert^2), \forall\, k\right\}$	$\mathrm{co}\left\{\frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right\}$	$\mathrm{co}\{0, \lambda_2, \ldots, \lambda_n\}$

Analogous criteria hold in the discrete time case.

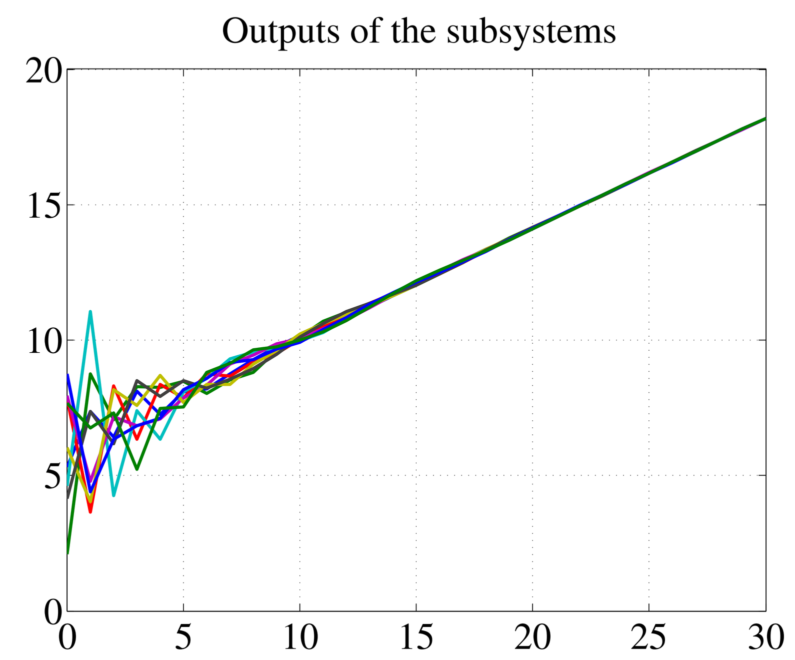
Clock synchronization

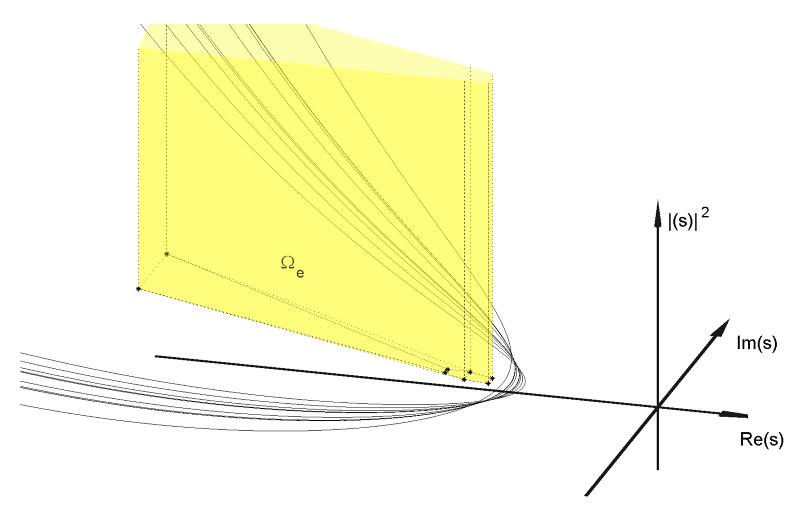
Model for the clock $\to$ double integrator plus control (common for all the clocks)

$$\begin{cases} x_k(t+1) = \begin{bmatrix} 1 & q_i \\ 0 & 1 \end{bmatrix} x_i(t) + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} u_k(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i(t) \\ u_k(t) = \sum_{j\in\mathcal{N}_k} \Gamma_{kj} y_j(t) \end{cases} \implies y_k(t) = H_k(t)u_k(t)$$

If $q_k = q + \varepsilon_k$, we obtain the following decomposition

$$N_0(z^{-1}) = \frac{z^{-2}(f_2 q - f_1) + f_1 z^{-1}}{(1-z^{-1})^2} \qquad N_k(z^{-1}) = \frac{f_2 \varepsilon_k z^{-2}}{(1-z^{-1})^2}.$$



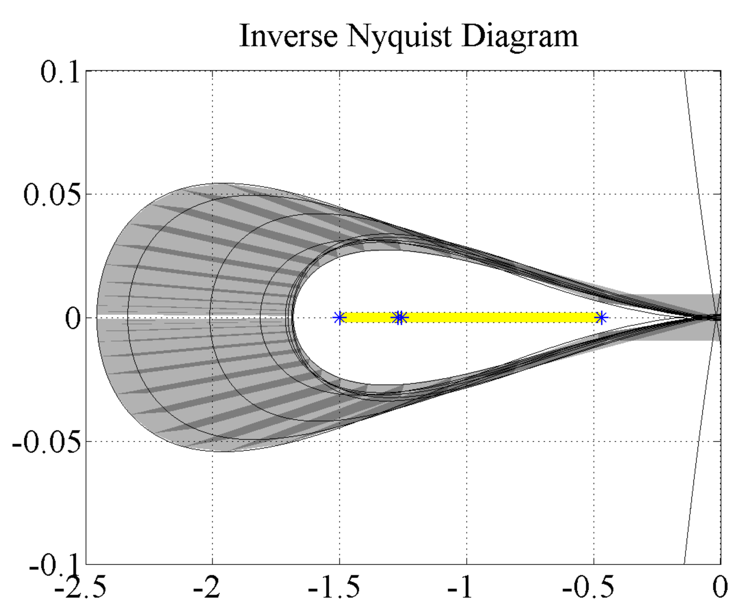


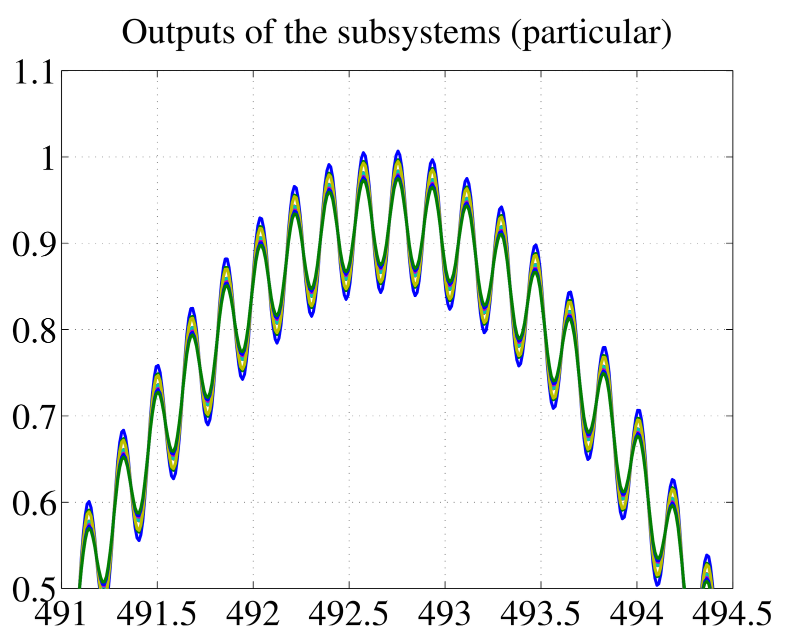
Result: conditions can be found on $\alpha > 1$ such that the **assumptions of the theorem are satisfied**!

Oscillators: a counterexample

We tried to synchronize a network in which $H_k(s) = N_0(s) + \frac{P_k(s)}{s^2+\omega_k^2}$ where N_0 has two imaginary poles.

The system shows only **partial synchronization**, maybe due to the "Nyquist" assumption.



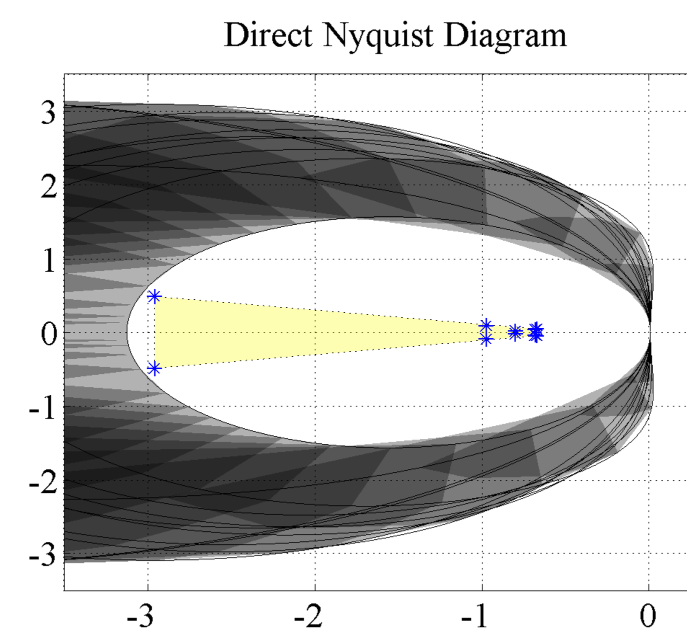


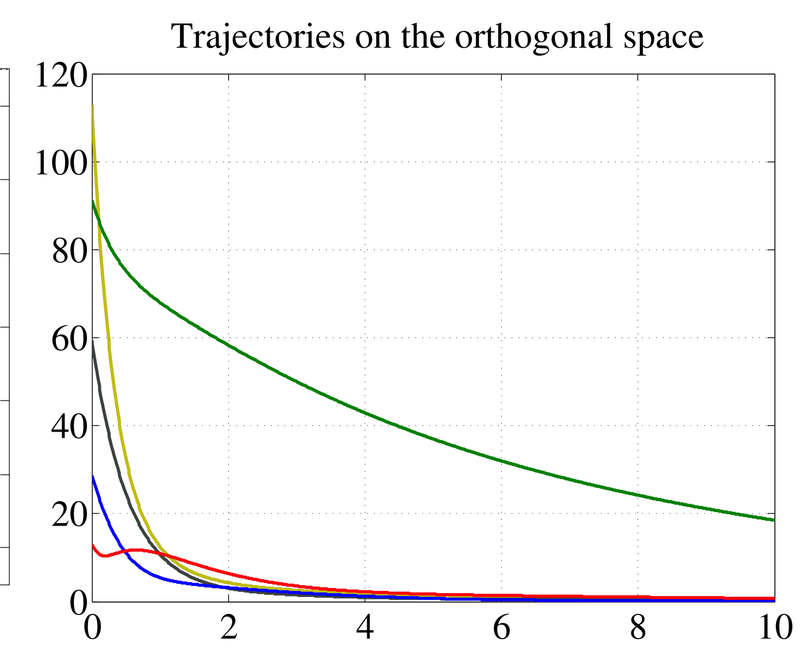
Unstable systems

The result can be applied in a **network of unstable systems** of the type

$$H_k(s) = \frac{1 + \Delta_k(s)}{s - \tau}$$

where $\tau > 0$ and $\Delta_k(s)$ are stable filters.





References

[1] E. Lovisari, U.T. Jönsson, A Nyquist criterion for synchronization in networks of heterogeneous linear systems In *Proceedings of 2nd IFAC NecSys*

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