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Necessary and Sufficient Conditions for Consensusability of Discrete-time Multi-agent systems

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Abstract

Under a common distributed control protocol, a necessary and sufficient condition for reaching a consensus of discrete-time multi-agent systems is given in terms of the agent dynamic and the **eigenratio** of an undirected communication graph. Here the **eigenratio** is defined as the ratio of the second smallest to the largest eigenvalues of the graph Laplacian matrix. The result is established by solving a discrete-time simultaneous stabilization problem. A lower bound of the optimal convergence rate to consensus, which is shown to be attainable for some special cases, is provided as well. It turns out that the intrinsic entropy rate of the agent dynamic and the eigenratio pose fundamental limitations on the consensus performance.

Consensusability on graphs

✓ Communication graph:

Undirected and fixed communication graph: $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$

Adjacency matrix $\mathcal{A}=[a_{ij}]$: $(i,j)\in\mathcal{E}\Leftrightarrow a_{ij}>0$

Eigenvalues of the graph Laplacian matrix is written as: $0=\lambda_1\le\lambda_2\le\cdots\le\lambda_N$.

✓Consider N agents with an identical discrete-time dynamical equation as follows:

$$X_i(k+1)=AX_i(k)+Bu_i(k), i=1,\ldots,N\cdots\cdots(1)$$

✓The following distributed control protocol is to be examined

$$u_i(k)=K\sum_{j=1}^{N}a_{ij}(X_j(k)-X_i(k))\in R,\cdots\cdots(2)$$

✓Given an undirected graph $\mathcal{G}$, the discrete-time multi-agent systems in (1) are said to be consensusable under the control protocol (2) if there exists a control gain K such that

$$\lim_{k\to\infty}\|X_i(k)-X_j(k)\|=0,\forall i,j.$$

✓Objective: Find necessary and sufficient conditions on the communication graph and the agent dynamic to enforce consensus and evaluate the convergence rate.

Necessary and sufficient condition on consensusability

Theorem 1: Given an undirected graph $\mathcal{G}$, the discrete-time multi-agent systems in (1) are consensusable under the control protocol (2) **if and only if** the following conditions hold:

❑ (A, B) is a stabilizable pair;

❑ Each agent cannot change too fast.

$$\prod_j|\lambda_j^u(A)|<\frac{1+\lambda_2/\lambda_N}{1-\lambda_2/\lambda_N},$$ $\lambda_j^u(A)$ denotes an unstable eigenvalue of A.

If all conditions hold, the control gain is designed as follows. Let P be a positive definite solution of the discrete-time Riccati inequality

$$P-A^TPA+\frac{A^TPBB^TPA}{B^TPB}>0,\ K=\frac{2}{\lambda_2+\lambda_N}\frac{B^TPA}{B^TPB}$$

Remark:

✓It contains the classical average consensus as a special case (Olfati-Saber and Murray, TAC04). In particular, the necessary and sufficient condition to reach an average consensus is that the communication graph is connected, which is equivalent to $\lambda_2>0$.

✓In contrast with the case of continuous-time agent dynamic, the intrinsic entropy rate of the agent dynamic poses a fundamental constraint on the **eigenratio** of the graph for consensusability. A larger eigenratio corresponds to a better synchronizability of the communication graph and allows a more unstable agent dynamic to reach a consensus. Note that in the continuous case, the information exchange is instantaneous.

✓ It can recover results on the continuous case by letting the sampling period be sufficiently small (Ma and Zhang, TAC10).

✓If the adjacency matrix of the graph is selected as a symmetric (0,1)-matrix, the **eigenratio** goes to one means that the graph becomes a complete graph. Under this case, the control protocol (2) is a centralized version. Thus, a stabilizable agent dynamic is sufficient for reaching a consensus of the multi-agent systems.

✓Note that adding an edge to the graph may lead to a smaller **eigenratio**. For example,

$$\mathcal{L}_1=\begin{bmatrix}3&-1&0&0&-1&-1\\-1&3&-1&-1&0&0\\0&-1&3&-1&0&-1\\0&-1&-1&3&-1&0\\-1&0&0&-1&3&-1\\-1&0&-1&0&-1&3\end{bmatrix},\ \mathcal{L}_2=\begin{bmatrix}4&-1&-1&0&-1&-1\\-1&3&-1&-1&0&0\\-1&-1&4&-1&0&-1\\0&-1&-1&3&-1&0\\-1&0&0&-1&3&-1\\-1&0&-1&0&-1&3\end{bmatrix}$$

$\mathcal{L}_2$, with an eigenratio 0.397 is formed by adding edge to the graph corresponding to $\mathcal{L}_1$, whose eigenratio is 0.4.

Performance of consensus protocol

✓Let $\overline{X}(k)=\frac{1}{N}\sum_{j=1}^{N}X_j(k)$ and $\delta_j(k)=X_j(k)-\overline{X}(k)$. Consensusability is equivalent to that

$$\lim_{k\to\infty}\|\delta_j(k)\|=0,\forall j.$$

✓Two measures of convergence speed:

asymptotic convergence factor

$$r_{asym}=\sup_{\delta(0)\neq0}\left(\frac{\|\delta(k)\|}{\|\delta(0)\|}\right)^{\frac{1}{k}}$$

Per-step convergence factor

$$r_{step}=\sup_{\delta(k)\neq0}\frac{\|\delta(k+1)\|}{\|\delta(k)\|}$$

Corollary 1: $r_{asym}=\max_{2\le j\le N}\rho(A-\lambda_jBK),\quad r_{step}=\max_{2\le j\le N}\|A-\lambda_jBK\|$

Theorem 2: Assume all the eigenvalues of *A* lie on or outside of the unit circle, the optimal asymptotic convergence factor is lower bounded by

$$r_{asym}^*=\inf_{K\in R^{1\times n}}\max_{2\le j\le N}\rho(A-\lambda_jBK)\ge|\det(A)|^{1/n}\left(\frac{1-\lambda_2/\lambda_N}{1+\lambda_2/\lambda_N}\right)^{\frac{1}{n}}$$

Remark: The lower bound is attainable under the following special cases.

✓The graph is a complete one and the adjacency matrix is a symmetric (0,1)-matrix.

✓The agent dynamic is an unstable scalar system.

✓It can also be approached for the discrete-time second-order consensus.

Second-order consensus

Agent dynamic is obtained by sampling a double-integrator system:

$$\begin{cases}x_i(k+1)=x_i(k)+hv_i(k)+\frac{1}{2}h^2u_i(k),\\ v_i(k+1)=v_i(k)+hu_i(k).\end{cases}$$

Theorem 3: The second-order multi-agent systems are consensusable under the control protocol (2) **if and only if** the communication graph is connected. Moreover, a control gain K solves the second-order consensus **if and only if**

$$K\in\{[\alpha,\beta]\,|\,\beta<\frac{2}{\lambda_Nh},0<\alpha<\frac{2\beta}{h}\}$$

Theorem 4: The optimal *asymptotic convergence factor* for the consensus of the second-order multi-agent systems is that

$$r_{asym}^*=\left(\frac{1-\lambda_2/\lambda_N}{1+\lambda_2/\lambda_N}\right)^{\frac{1}{2}}.$$

Moreover, the following control gain leads to the optimal *asymptotic convergence factor*:

$$K^*=[\frac{1-(r_{asym}^*)^2}{h^2\lambda_N},\ \frac{3+(r_{asym}^*)^2}{2h\lambda_N}].$$

An example

The desired formation is specified as the vertices of a regular parallelogram.

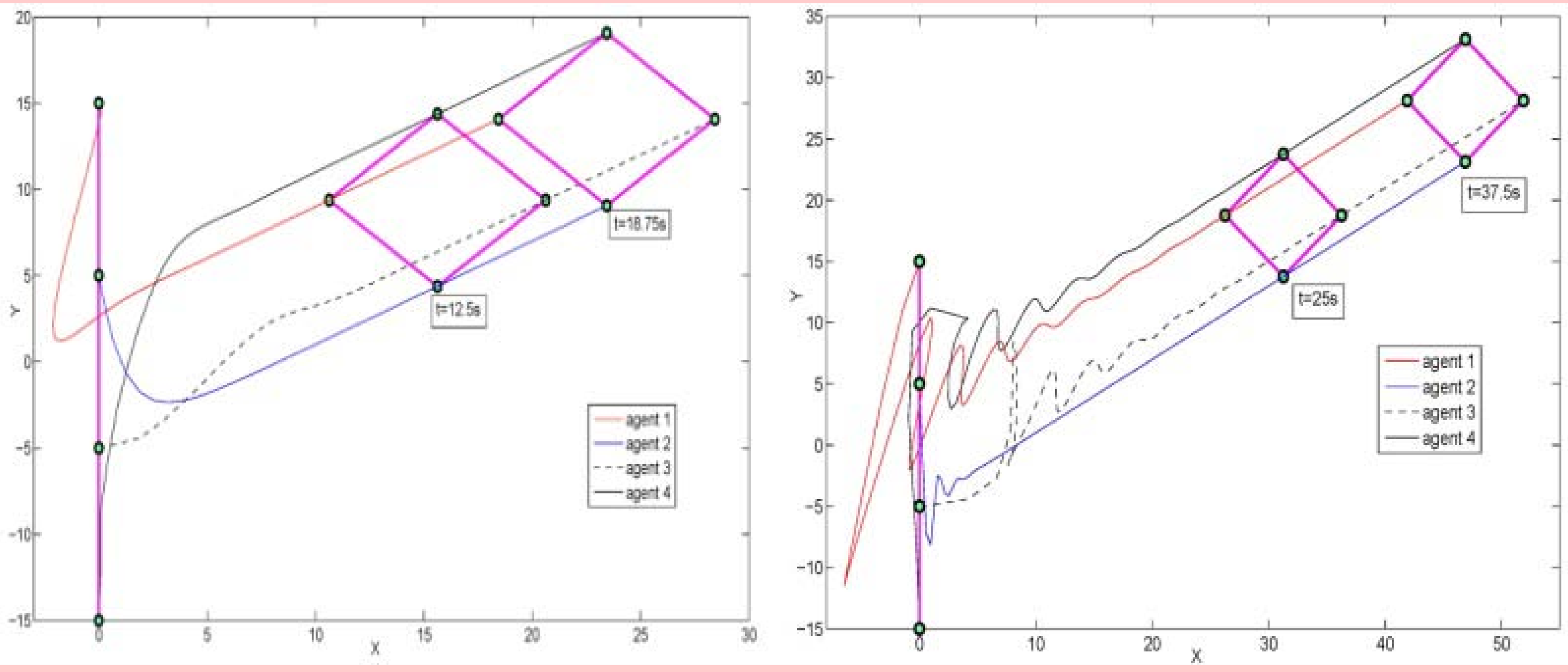

Fig 1. Fastest convergence versus convergence

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