Optimal Control of Networked Control Systems with Uncertain Time-Varying Transmission Delay

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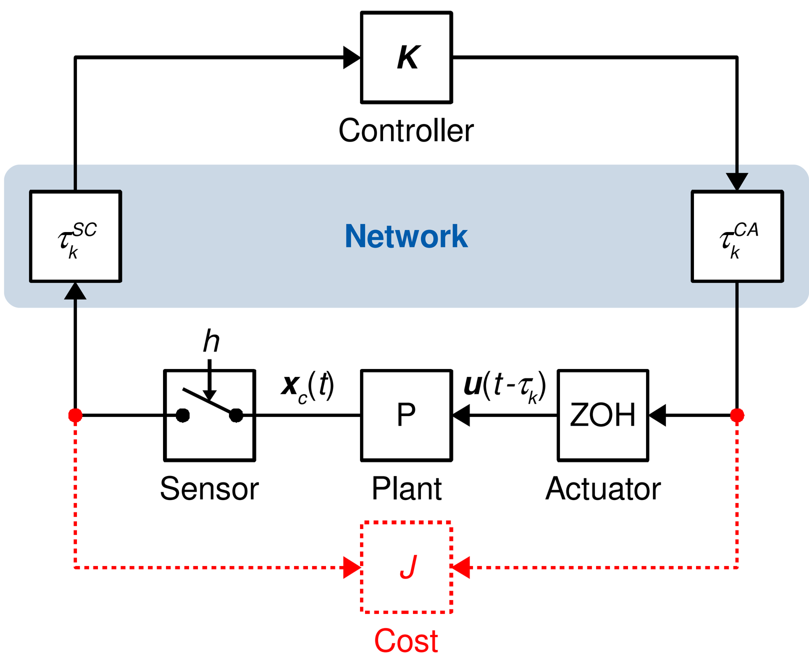
Introduction

Motivation

- Control systems are increasingly realized as networked control systems (NCS)
- Communication leads to uncertain time-varying transmission delay $\tau_k = \tau_k^{CA} + \tau_k^{SC}$
- Discrete-time models of NCS depend nonlinearly on the uncertainty in τ_k
  - Standard robust control theory not applicable
  - Polytopic overapproximation
  - Previous work primarily focuses on stability



Contributions

- Polytopic overapproximation of both the plant model and the cost function
- Optimal control design based on a parameter-dependent Lyapunov function

Modeling

System Model and Cost Function

- Continuous-time system

$$\dot{\boldsymbol{x}}_c(t) = \boldsymbol{A}_c\boldsymbol{x}_c(t) + \boldsymbol{B}_c\boldsymbol{u}(t-\tau_k)$$

  - Time-varying input delay τ_k represents the transmission delay
  - Discretized for constant sampling period h / uncertain time-varying time delay $\tau_k \in \mathcal{I} = [\underline{\tau}, \overline{\tau}]$ with $\underline{\tau} = \tau_{\text{nom}} - \tau_\Delta$, $\overline{\tau} = \tau_{\text{nom}} + \tau_\Delta$, $\tau_\Delta \geq 0$ and $\overline{\tau} \leq h$ using ZOH

- Discrete-time system

$$\begin{pmatrix} \boldsymbol{x}_c(k+1) \\ \boldsymbol{u}(k) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}(h) & \boldsymbol{\Gamma}(h)-\boldsymbol{\Gamma}(h-\tau_k) \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \underbrace{\begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \end{pmatrix}}_{\boldsymbol{x}(k)} + \begin{pmatrix} \boldsymbol{\Gamma}(h-\tau_k) \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{u}(k) \qquad (1)$$

with

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}_c t}, \qquad \boldsymbol{\Gamma}(t) = \int_0^t \boldsymbol{\Phi}(t)ds\boldsymbol{B}_c$$

- Continuous- and discrete-time cost function

$$J = \int_0^\infty \begin{pmatrix} \boldsymbol{x}_c(t) \\ \boldsymbol{u}(t-\tau(t)) \end{pmatrix}^T \begin{pmatrix} \boldsymbol{Q}_{1c} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_{2c} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_c(t) \\ \boldsymbol{u}(t-\tau(t)) \end{pmatrix} dt = \sum_{k=0}^\infty \begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \\ \boldsymbol{u}(k) \end{pmatrix}^T \boldsymbol{Q}(\tau_k) \begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \\ \boldsymbol{u}(k) \end{pmatrix} \qquad (2)$$

with

$$\boldsymbol{Q}(\tau_k) = \begin{pmatrix} \int_0^h \boldsymbol{\Phi}^T(t)\boldsymbol{Q}_{1c}\boldsymbol{\Phi}(t)dt & \int_0^{\tau_k} \boldsymbol{\Phi}^T(t)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}(t)dt & \int_{\tau_k}^h \boldsymbol{\Phi}^T(t)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}(t-\tau_k)dt \\ * & \int_0^{\tau_k} \boldsymbol{\Gamma}^T(t)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}(t)+\boldsymbol{Q}_{2c}dt & \boldsymbol{0} \\ * & * & \int_{\tau_k}^h \boldsymbol{\Gamma}^T(t-\tau_k)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}(t-\tau_k)+\boldsymbol{Q}_{2c}dt \end{pmatrix}$$

> *Problem 1.* For the discrete-time system (1) find a control sequence $\boldsymbol{u}(k)$ such that the discrete-time cost function (2) is robustly minimized for all sequences $\tau_k \in \mathcal{I}$, i.e.
>
> $$\min_{\boldsymbol{u}(k)} \max_{\tau_k \in \mathcal{I}} J \quad \text{subject to (1).}$$

- Uncertain $\tau_k \in \mathcal{I}$ affects (1) and (2) in a nonlinear manner ⇒ Use polytopic overapproximation

Polytopic Formulation

- Taylor series expansion

$$\boldsymbol{\Phi}(t) = \sum_{q=0}^\infty \frac{\boldsymbol{A}_c^q}{q!}t^q = \underbrace{\sum_{q=0}^{M_\Phi} \frac{\boldsymbol{A}_c^q}{q!}t^q}_{\boldsymbol{\Phi}_{\text{TE}}(t,M_\Phi)} + \Delta\boldsymbol{\Phi}, \qquad \boldsymbol{\Gamma}(t) = \int_0^t \sum_{q=0}^\infty \frac{\boldsymbol{A}_c^q}{q!}s^q ds\boldsymbol{B}_c = \underbrace{\sum_{q=1}^{M_\Gamma+1} \frac{\boldsymbol{A}_c^{q-1}}{q!}t^q\boldsymbol{B}_c}_{\boldsymbol{\Gamma}_{\text{TE}}(t,M_\Gamma)} + \Delta\boldsymbol{\Gamma}$$

$$\boldsymbol{Q}(\tau_k) = \underbrace{\begin{pmatrix} \boldsymbol{Q}_{11} & \int_0^{\tau_k} \boldsymbol{\Phi}_{\text{TE}}^T(t,M_{12}^a)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}_{\text{TE}}(t,M_{12}^b)dt & \int_{\tau_k}^h \boldsymbol{\Phi}_{\text{TE}}^T(t,M_{13}^a)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}_{\text{TE}}(t-\tau_k,M_{13}^b)dt \\ * & \int_0^{\tau_k} \boldsymbol{\Gamma}_{\text{TE}}^T(t,M_{22}^a)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}_{\text{TE}}(t,M_{22}^b)+\boldsymbol{Q}_{2c}dt & \boldsymbol{0} \\ * & * & \int_{\tau_k}^h \boldsymbol{\Gamma}_{\text{TE}}^T(t-\tau_k,M_{33}^a)\boldsymbol{Q}_{1c}\boldsymbol{\Gamma}_{\text{TE}}(t-\tau_k,M_{33}^b)+\boldsymbol{Q}_{2c}dt \end{pmatrix}}_{\boldsymbol{Q}_{\text{TE}}(\tau_k,M_{12}^a,M_{12}^b,\ldots)} + \Delta\boldsymbol{Q}$$

- Truncation parts are matrix polynomials
- Polynomial orders
  - multiplication ⇔ addition of polynomial orders
  - integration ⇔ increase by one
  - Truncation orders can be chosen differently but polynomial orders must be identical

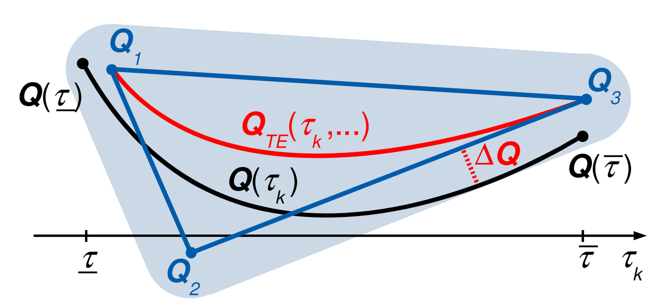
Matrix	Polynomial Order	Truncation Orders
$\boldsymbol{\Gamma}$	$M_\Gamma+1$	$M_\Gamma = 2M+2$
$\boldsymbol{Q}_{12}$	$M_{12}^a+M_{12}^b+2$	$M_{12}^a = M+1$, $M_{12}^b = M$
$\boldsymbol{Q}_{13}$	$M_{13}^a+M_{13}^b+2$	$M_{13}^a = M+1$, $M_{13}^b = M$
$\boldsymbol{Q}_{22}$	$M_{22}^a+M_{22}^b+3$	$M_{22}^a = M_{22}^b = M$
$\boldsymbol{Q}_{33}$	$M_{33}^a+M_{33}^b+3$	$M_{33}^a = M_{33}^b = M$

Polynomial order $\tilde{M} = 2M+3$

- The matrix polynomials can be enveloped by polytopes

$$\boldsymbol{\Gamma}_{\text{TE}}(\tau_k, M_\Gamma) = \sum_{i=0}^{\tilde{M}} \mu_i(\tau_k)\boldsymbol{\Gamma}_i$$

$$\boldsymbol{Q}_{\text{TE}}(\tau_k, M_{12}^a, M_{12}^b, \ldots) = \sum_{i=0}^{\tilde{M}} \mu_i(\tau_k)\boldsymbol{Q}_i$$



- Discrete-time system with polytopic and additive norm-bounded uncertainty

$$\boldsymbol{x}(k+1) = \left(\sum_{i=0}^{\tilde{M}} \mu_i(\tau_k) \underbrace{\begin{pmatrix} \boldsymbol{\Phi}(h) & \boldsymbol{\Gamma}(h)-\boldsymbol{\Gamma}_i \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{\boldsymbol{A}_i} + \underbrace{\begin{pmatrix} \boldsymbol{0} & -\Delta\boldsymbol{\Gamma} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{\Delta\boldsymbol{A}} \right) \boldsymbol{x}(k) + \left(\sum_{i=0}^{\tilde{M}} \mu_i(\tau_k) \underbrace{\begin{pmatrix} \boldsymbol{\Gamma}_i \\ \boldsymbol{I} \end{pmatrix}}_{\boldsymbol{B}_i} + \underbrace{\begin{pmatrix} \Delta\boldsymbol{\Gamma} \\ \boldsymbol{0} \end{pmatrix}}_{\Delta\boldsymbol{B}} \right) \boldsymbol{u}(k)$$

- Discrete-time cost function with polytopic and additive norm-bounded uncertainty

$$J = \sum_{k=0}^\infty \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{u}(k) \end{pmatrix}^T \left(\sum_{i=0}^{\tilde{M}} \mu_i(\tau_k)\boldsymbol{Q}_i + \Delta\boldsymbol{Q} \right) \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{u}(k) \end{pmatrix}$$

Control

Closed-Loop Formulation

- Full state feedback control law

$$\boldsymbol{u}(k) = \underbrace{\begin{pmatrix} \boldsymbol{K}_x & \boldsymbol{K}_u \end{pmatrix}}_{\boldsymbol{K}} \begin{pmatrix} \boldsymbol{x}_c(k) \\ \boldsymbol{u}(k-1) \end{pmatrix}$$

- Discrete-time closed-loop system

$$\boldsymbol{x}(k+1) = \left(\sum_{i=0}^{\tilde{M}} \mu_i(\tau_k) \underbrace{\begin{pmatrix} \boldsymbol{\Phi}(h)+\boldsymbol{\Gamma}_i\boldsymbol{K}_x & \boldsymbol{\Gamma}(h)+\boldsymbol{\Gamma}_i(\boldsymbol{K}_u-\boldsymbol{I}) \\ \boldsymbol{K}_x & \boldsymbol{K}_u \end{pmatrix}}_{\tilde{\boldsymbol{A}}_i} + \underbrace{\begin{pmatrix} \Delta\boldsymbol{\Gamma}\boldsymbol{K}_x & \Delta\boldsymbol{\Gamma}(\boldsymbol{K}_u-\boldsymbol{I}) \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}}_{\Delta\tilde{\boldsymbol{A}}} \right) \boldsymbol{x}(k) \qquad (3)$$

- Discrete-time closed-loop cost function

$$J = \sum_{k=0}^\infty \boldsymbol{x}^T(k) \left(\underbrace{\sum_{i=0}^{\tilde{M}} \mu_i(\tau_k) \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{pmatrix}^T \boldsymbol{Q}_i \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{pmatrix}}_{\tilde{\mathcal{Q}}(\tau_k)} + \underbrace{\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{pmatrix}^T \Delta\boldsymbol{Q} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{pmatrix}}_{\Delta\tilde{\mathcal{Q}}} \right) \boldsymbol{x}(k) \qquad (4)$$

- The additive norm-bounded uncertainty can be addressed by different approaches and will not be considered in the following.

> *Problem 2.* For the discrete-time closed-loop system (3) find a feedback gain $\boldsymbol{K}$ such that the discrete-time closed-loop cost function (4) is robustly minimized for all sequences $\tau_k \in \mathcal{I}$, i.e.
>
> $$\min_{\boldsymbol{K}} \max_{\tau_k \in \mathcal{I}} J \quad \text{subject to (3).}$$

- Problem 2 is computationally intractable ⇒ Minimize an upper bound instead

Upper Bound on the Cost Function

- For a bounded value of the cost function (4) the quadratic function

$$V(\boldsymbol{x}(k), k) = \boldsymbol{x}^T(k)\mathcal{P}(\tau_k)\boldsymbol{x}(k) \qquad \text{with } \mathcal{P}(\tau_k) = \sum_{i=0}^{\tilde{M}} \mu_i(\tau_k)\boldsymbol{P}_i, \quad \boldsymbol{P}_i \succ \boldsymbol{0}$$

satisfying

$$V(\boldsymbol{x}(k+1), k+1) - V(\boldsymbol{x}(k), k) < -\boldsymbol{x}^T(k)\tilde{\mathcal{Q}}(\tau_k)\boldsymbol{x}(k) \qquad (5)$$

for all $\tau_k \in \mathcal{I}$ and $\boldsymbol{x}(k) \neq \boldsymbol{0}$ upper bounds the cost function, i.e.

$$\max_{\tau_k \in \mathcal{I}} J < V(\boldsymbol{x}(0), 0) \quad \Leftrightarrow \quad \max_{\tau_k \in \mathcal{I}} J < \boldsymbol{x}^T(0)\mathcal{P}(\tau_0)\boldsymbol{x}(0),$$

which follows by summing (5) over $k = 0, \ldots, \infty$

- The maximum cost degradation w.r.t. the nominal case is given by

$$\mathcal{P}(\tau_0) \prec \gamma\boldsymbol{P}_{\text{nom}} \qquad (6)$$

where $\boldsymbol{P}_{\text{nom}}$ is the symmetric and positive definite solution of the ARE underlying the LQR problem related to a constant sampling period h and a nominal transmission delay τ_{nom}

- Problem 2 can be redefined to: $\min_{\boldsymbol{K},\boldsymbol{P}_i} \gamma$ subject to (5) and (6)

Control Design

> *Theorem 1.* An upper bound on γ is obtained from the LMI optimization problem
>
> $$\max \hat{\gamma} \quad \text{subject to} \quad \hat{\gamma} > 0, \quad \boldsymbol{Z}_i \succ \hat{\gamma}\boldsymbol{P}_{\text{nom}}^{-1}, \quad \begin{pmatrix} \boldsymbol{G}^T+\boldsymbol{G}-\boldsymbol{Z}_i & * & * \\ \boldsymbol{Q}_i^{\frac{1}{2}} \begin{pmatrix} \boldsymbol{G} \\ \boldsymbol{W} \end{pmatrix} & \boldsymbol{I} & * \\ \boldsymbol{A}_i\boldsymbol{G}+\boldsymbol{B}_i\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{Z}_j \end{pmatrix} \succ \boldsymbol{0}$$
>
> for all $i, j = 1, \ldots, \tilde{M}$ in the variables $\boldsymbol{G}, \boldsymbol{W}, \boldsymbol{Z}_i$ and $\hat{\gamma}$. The corresponding robust feedback matrix is $\boldsymbol{K} = \boldsymbol{W}\boldsymbol{G}^{-1}$ and the minimum upper bound results from $\gamma = 1/\hat{\gamma}$.

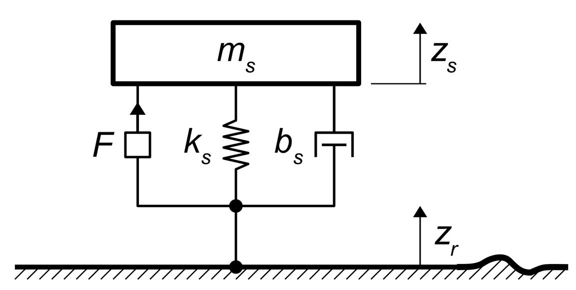
Example

- Networked control of an active suspension system

$$\dot{\boldsymbol{x}}_c(t) = \begin{pmatrix} 0 & 1 \\ -k_s/m_s & -b_s/m_s \end{pmatrix} \boldsymbol{x}_c(t) + \begin{pmatrix} 0 \\ 1/m_s \end{pmatrix} u(t-\tau) = \begin{pmatrix} \ddot{z}_s(t) \\ \dot{z}_s(t)-\dot{z}_r(t) \end{pmatrix}$$

$$\boldsymbol{y}(t) = \begin{pmatrix} -k_s/m_s & -b_s/m_s \\ 1 & 0 \end{pmatrix} \boldsymbol{x}_c(t) + \begin{pmatrix} 1/m_s \\ 0 \end{pmatrix} u(t-\tau) = \begin{pmatrix} \ddot{z}_s(t) \\ z_s(t)-z_r(t) \end{pmatrix}$$

with $\dot{z}_r(0) = 0$, $m_s = 315\,\text{kg}$, $k_s = 29.5\,\frac{\text{kN}}{\text{m}}$, $b_s = 1.5\,\frac{\text{kNs}}{\text{m}}$

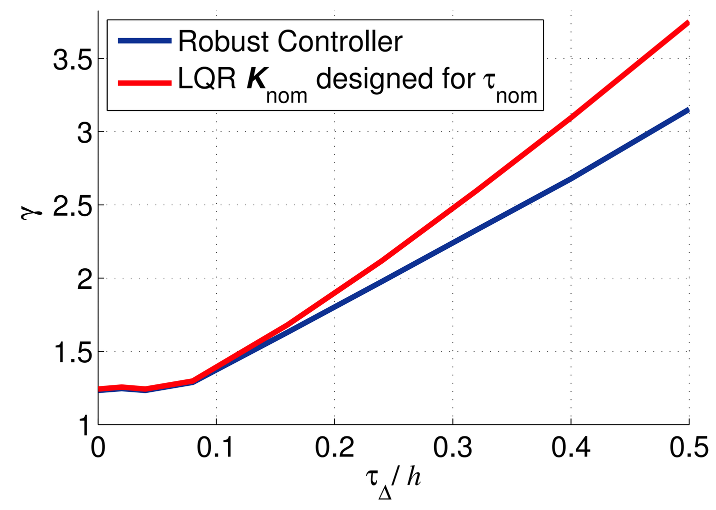


- Cost function

$$J = \int_0^\infty q_1\ddot{z}_s(t)^2 + q_2(z_s(t)-z_r(t))^2 + rF(t-\tau)^2 dt$$

with $q_1 = 1000$, $q_2 = 10$, $r = 0.02$

- Timing and design parameters
  - Constant sampling period $h = 25\,\text{ms}$
  - Nominal transmission delay $\tau_{\text{nom}} = 12.5\,\text{ms}$
  - Truncation order $M = 4$
- Cost degradation for LQR follows from Theorem 1 by setting $\boldsymbol{W} = \boldsymbol{G}\boldsymbol{K}_{\text{nom}}$



Conclusions

- Modeling and optimal control design method allowing to assess and handle the effect of an uncertain time-varying transmission delay on the control performance in a quantitative way
- Formulation for uncertain time-varying sampling period / switched interval uncertainty possible

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