The Asymptotic Consensus Problem on Convex Metric Spaces

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MOTIVATION

Two agreement/consensus algorithms

- Initial values $x_i(0) \in \mathbb{R}^n$
- Iterative algorithm

$$x_i(k+1) = \sum_{i \in \mathcal{N}_i} w_{ij} x_j(k),$$

with $\sum_{i \in \mathcal{N}_i} w_{ij} = 1$ and $w_{ij} > 0$.

- Geometric interpretation

$$x_i(k+1) \in conv(x_j(k), j \in \mathcal{N}_j)$$

- Initial values $X_i(0) \in \{1, \dots, s\}$
- Iterative algorithm

$$X_i(k+1) = \sum_{i \in \mathcal{N}_i} \mathbb{1}_{\{\theta_i = j\}} X_j(k),$$

where $Pr(\theta_i = j) = w_{ij}$.

- Geometric interpretation

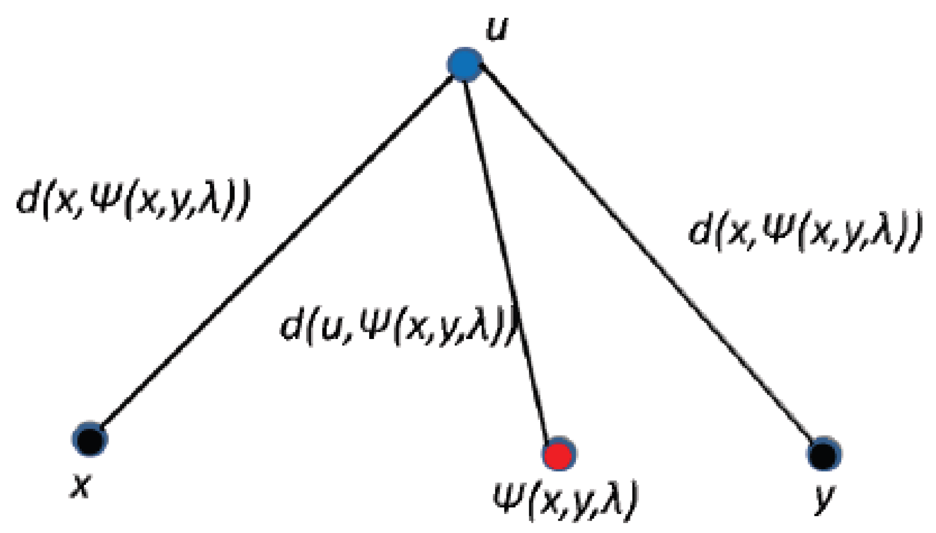
$$X_i(k+1) \in ?$$

CONVEX METRIC SPACES

- A metric space $(\mathcal{X}, d)$ together with a *convex structure* $\psi : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ such that

$$d(u, \psi(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y), \ \forall x,y,u \in \mathcal{X} \text{ and } \forall \lambda \in [0,1],$$

form a convex metric space



Examples of Convex Metric Spaces

- $\mathcal{X} = \mathbb{R}^n$, $d(x,y) = \|x-y\|$ with the convex structure $\psi(x,y,\lambda) = \lambda x + (1-\lambda) y$
- $\mathcal{X} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$ with the convex structure

$$\psi(I_i, I_j, \lambda) = [\lambda a_i + (1-\lambda) a_j, \lambda b_i + (1-\lambda) b_j],$$

for all $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and $\lambda \in [0,1]$ and metric (Hausdorff distance)

$$d(I_i, I_j) = \sup_{a \in I} \{ | \inf_{b \in I_i} \{|a-b|\} - \inf_{c \in I_j} \{|a-c|\} | \}$$

Convex sets

- A nonempty subset $A \subset \mathcal{X}$ is said to be *convex* if $\psi(x,y,\lambda) \in A$, $\forall x,y \in A$ and $\forall \lambda \in [0,1]$
- *Convex hull* of a set A

$$conv(A) = \bigcup_{m=0}^{\infty} A_m,$$

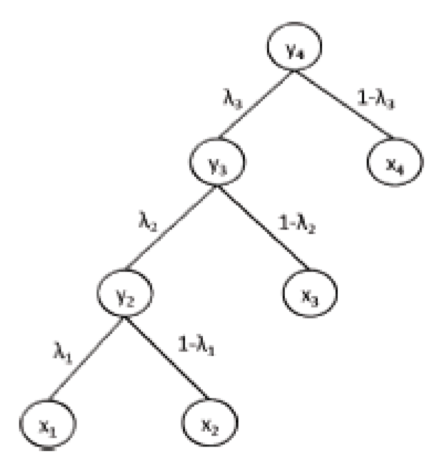
where $A_{m+1} = \{\psi(x,y,\lambda) \mid \forall x,y \in A_m, \lambda \in [0,1]\}$ and $A_0 = A$

- Generating points that belong to $conv(A)$, with $A = \{x_1, \dots, x_n\}$

$y_{i+1} = \psi(y_i, x_{i+1}, \lambda_i)$ for $i = 1 \dots n-1$

with $y_1 = x_1$

$$y_n \in conv(A)$$



MODEL

- Group of N agents organized in a network modeled by a directed graph $G = (V, E)$
- Agents' objective is to **agree** on a quantity belonging to a convex metric space $(\mathcal{X}, d, \psi)$

AGREEMENT ALGORITHM

- Connectivity neighborhood $\mathcal{N}_i = \{j \mid (i,j) \in E\}$
- Local information set $A_i(k) = \{x_i(k), j \in \mathcal{N}_i\}$.

Theorem 3.1 Under the assumption that G is connected, if

$$x_i(k+1) \in conv(A_i(k)), \ \forall i,$$

then

$$\lim_{k \to \infty} d(x_i(k), x_j(k)) = 0, \ \forall i,j, \ i \neq j.$$

Corollary 3.1 Under the assumption that G is connected, if

$$x_i(k+1) \in conv(A_i(k)), \ \forall i,$$

then there exits $x^* \in \mathcal{X}$ such that

$$lim_{k \to \infty} d(x_i(k), x^*) = 0, \ \forall i$$

GEOMETRY OF THE CONSENSUS OF OPINION ALGORITHM

- Let $\mathcal{X}$ be the space of discrete random variables taking values in $\{1, 2, \dots, s\}$
- Consider the metric $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$

$$d(X,Y) = E[\rho(X,Y)] = E[\mathbb{1}_{\{X \neq Y\}}] = Pr(X \neq Y),$$

where $\rho : \mathbb{R} \times \mathbb{R} \to \{0,1\}$ is the discrete metric, i.e.

$$\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

- Consider the mapping $\psi : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$, give by

$$\psi(X_1, X_2, \lambda) = \mathbb{1}_{\{\theta=1\}} X_1 + \mathbb{1}_{\{\theta=2\}} X_2, \ \forall X_1, X_2 \in \mathcal{X}, \lambda \in [0,1],$$

where $Pr(\theta = 1) = \lambda$ and $Pr(\theta = 2) = 1 - \lambda$

- $(\mathcal{X}, d, \psi)$ is a *convex metric space*

Consensus of opinion algorithm - Geometric interpretation

- Convex hull of a finite set $A = \{X_1, X_2, \dots, X_n\}$

$$conv(A) = \left\{ Z \in \mathcal{X} \mid Z = \sum_{i=1}^{n} \mathbb{1}_{\{\theta=i\}} X_i, \ \forall w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\},$$

where $Pr(\theta = i) = w_i$

- Algorithm for reaching consensus

$$X_i(k+1) \in conv(A_i(k)),$$

where $A_i(k) = \{X_i(k), j \in \mathcal{N}_i\}$

- OR EQUIVALENTLY

$$X_i(k+1) = \sum_{i \in \mathcal{N}_i} \mathbb{1}_{\{\theta_i = j\}} X_j(k),$$

where $Pr(\theta_i = j) = w_{ij}$ with $\sum_{j \in \mathcal{N}_i} w_{ij} = 1$