On the Minimal Average Data-Rate that Guarantees a Given Closed Loop Performance Level

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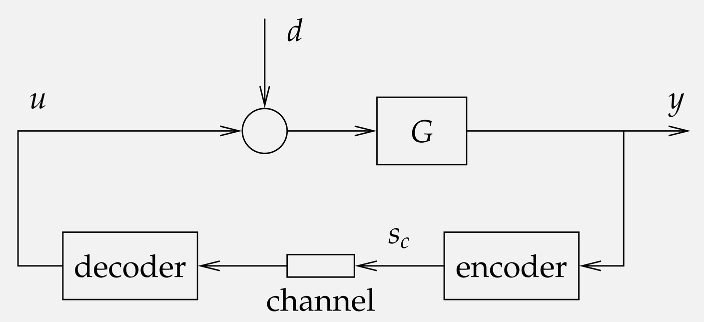
Summary

- **Setup:** SISO LTI plants, error-free digital channels.
- **Aim:** Finding the minimum average data-rate $\mathcal{R}_D$ that allows one to attain a given performance level D.
- **Results:** By focusing on a class of control and source coding schemes, and using a constructive approach that does not rely on asymptotic approximations, we find:
  - Upper and lower bounds on $\mathcal{R}_D$.
  - A specific control and coding scheme that achieves average data-rates between our bounds.

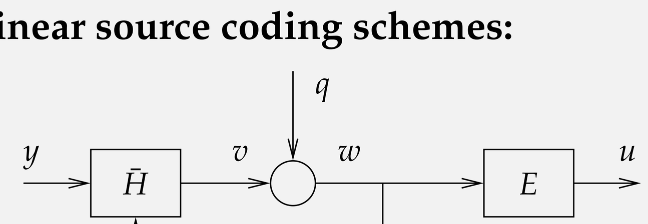
Setup

- **Networked control system (NCS) scheme:**



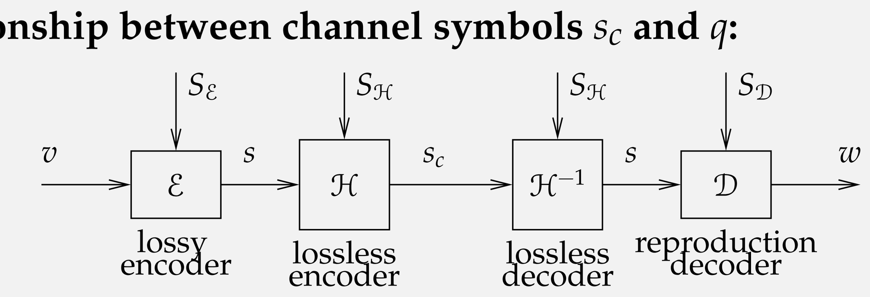
  - G is a SISO LTI plant with Gaussian initial state x_o.
  - d is a 2nd order stationary Gaussian sequence.
  - The channel is a error-free digital channel with input s_c (binary words).
  - Decoder and encoder are causal (they embed a controller).

- **The class of linear source coding schemes:**



  - $\bar{H}$ and E are proper LTI systems.
  - q is an i.i.d. sequence, independent of (x_o, d) (e.g., noise in subtractively dithered quantizers [Zamir & Feder, 1992]).

- **Relationship between channel symbols s_c and q:**



  - $\mathcal{E}, \mathcal{D}, \mathcal{H}$ and $\mathcal{H}^{-1}$ are causal.
  - $S_{(\cdot)}$ is side information that is available at $(\cdot)$ (added for generality).
  - $S_\mathcal{D}$ is independent of (x_o, d).

- **Definition of average data-rate:**
  Let $R(i)$ be the expected length of the binary word $s_c(i)$. Then
  $$\mathcal{R} \triangleq \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} R(i).$$
  denotes the average data-rate across the channel.

Main problem

Denote by $D_{\inf}$ the minimum stationary variance of y that is achievable when $q = 0$. For any performance level $D \in (D_{\inf}, \infty)$, find
$$\mathcal{R}_D \triangleq \inf_{\sigma_y^2 \leqslant D} \mathcal{R},$$
where σ_y^2 is the stationary variance of y, and the search is over:

- All causal $\mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{H}^{-1}$ such that q is as above.
- All causal filters $\bar{H}$ and E that internally stabilize the NCS.

Finding $\mathcal{R}_D$ is akin to solving a causal rate-distortion problem.

Key first step

The next theorem relates $\mathcal{R}$ to 2nd order statistics of w and q:

Theorem (Silva et al, 2010)

If $\bar{H}$ and E stabilize (internally) the NCS, then
$$\mathcal{R} \geqslant I_\infty(v \to w) \geqslant \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega,$$
where

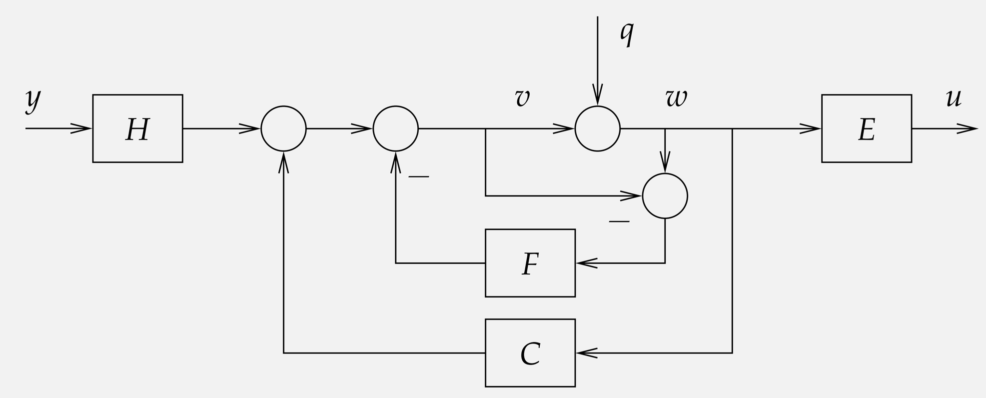
- *$I_\infty(v \to w)$ denotes the directed (mutual) information rate [Massey, 1990] between v and w,*
- *S_w is the stationary power spectral density of w, and*
- *σ_q^2 is the variance of q.*

Moreover, equality holds in the last inequality if and only if q is Gaussian

By using the proposed class of coding schemes, one can focus on the spectral properties of w and q to obtain lower bounds on $\mathcal{R}$.

Key second step

The optimal design of $\bar{H}$ and E is a non-convex problem [Rotkowitz & Lall, 2006]. We thus introduce an over-parametrization of $\bar{H}$:



$$\mathcal{S} \triangleq \left\{(H, C, F, E) \in \mathcal{R}_p^4 : \text{the resulting LTI system is internally stable and well-posed}\right\}.$$

Lemma

$$\inf_{\substack{\sigma_q^2\in\mathbb{R}^+\\(H,C,F,E)\in\mathcal{S}\\ \sigma_y^2\leqslant D}} \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{S_w}{\sigma_q^2}d\omega = \inf_{\substack{\sigma_q^2\in\mathbb{R}^+\\(H,C,F,E)\in\mathcal{S}\\ \sigma_y^2\leqslant D}} \frac{1}{2}\ln(1+\gamma)$$
where
$$\gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2},$$
and σ_v^2 is the stationary variance of v; γ is the SNR of the source coding scheme.

The over-parametrization allows one to:

- Carry out the design of $\bar{H}$ and E in an optimal fashion.
- Focus on the SNR γ in order to get lower bounds on $\mathcal{R}$.

Solving an auxiliary problem

Define
$$\gamma_{\inf} \triangleq \left(\prod_{i=1}^{n_p}|p_i|^2\right) - 1, \tag{1}$$
where p_i is the i^{th} unstable pole of G. Find, for a given $\Gamma \in (\gamma_{\inf}, \infty)$,
$$\left[\sigma_y^2\right]_\Gamma \triangleq \inf_{\sigma_q^2\in\mathbb{R}^+,\ \substack{(H,C,F,E)\in\mathcal{S}\\ \gamma\leqslant\Gamma}} \sigma_y^2,$$

Theorem

If $EH \neq 0$ at the optimum, then
$$\left[\sigma_y^2\right]_\Gamma \geqslant \inf_{\substack{f\in\mathcal{M}\\ Q\in\mathcal{RH}_\infty}} J_\Gamma(f, Q) \triangleq J_{\Gamma,\inf}, \tag{*}$$
where
$$\mathcal{M} \triangleq \left\{ f : \mathbb{D} \to \mathbb{R}_0^+ : \|f\|_2^2 < \Gamma + 1, \text{ and } \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f d\omega \geqslant \sum_{i=1}^{n_p}\ln|p_i| \right\},$$
and
$$J_\Gamma(f, Q) \triangleq \left\|W_Q + \Omega_x\right\|_2^2 + \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|W_Q\right| f d\omega\right)^2}{\Gamma + 1 - \|f\|_2^2},$$
where W_Q is affine in Q, and Ω_x depend on G and the PSD of d.

Moreover, the optimization problem on the RHS of (*) is convex.

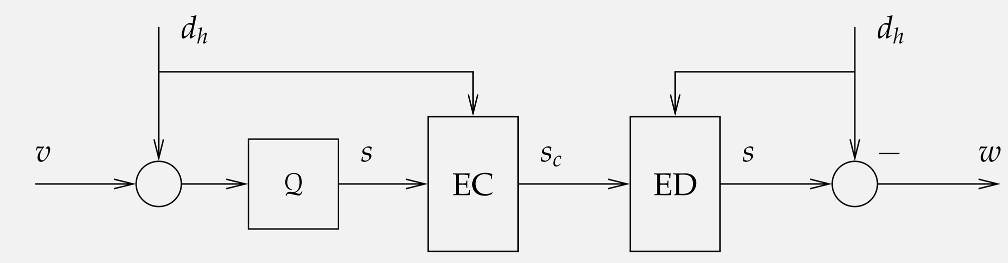
Theorem

If $EH \neq 0$ at the optimum, then:

- *$\left[\sigma_y^2\right]_\Gamma = J_{\Gamma,\inf}$.*
- *For any $\epsilon > 0$, there exists $\delta > 0$, $Q_\delta \in \mathcal{RH}_\infty$ and $f_\delta \in \mathcal{M}$ satisfying*
  $$J_\Gamma(f_\delta, Q_\delta) \leqslant J_{\Gamma,\inf} + \delta,$$
  that allow one to build $(H, C, F, E) \in \mathcal{S}$ and to find $\sigma_q^2 \in \mathbb{R}_0^+$ (all functions of Q_δ and f_δ) such that
  $$\sigma_y^2 \leqslant J_{\Gamma,\inf} + \epsilon, \quad \gamma = \Gamma.$$

Building the noise source q

We use an entropy coded dithered quantizer (ECDQ) as the link between v and w [Zamir & Feder, 1995]:



- $\mathcal{Q}$ is a uniform quantizer with step size $\Delta > 0$.
- d_h is a dither signal (available at encoder and decoder sides).
- EC-ED is an entropy-coder entropy-decoder (EC-ED) pair that works conditioned upon the current dither value $d_h(k)$ [Cover and Thomas, 2006].

Theorem (Zamir & Feder, 1995, 1996, 2008)

If $\Delta < \infty$ and d_h is i.i.d., independent of (x_o, d), and uniformly distributed on $(-\Delta/2, \Delta/2)$, then:

- *The quantization noise $q \triangleq w - v$ is i.i.d., independent of (x_o, d), and uniformly distributed on $(-\Delta/2, \Delta/2)$.*
- *There exists an EC-ED pair such that*
  $$\frac{1}{2}\ln(1+\gamma) \leqslant \mathcal{R} < \frac{1}{2}\ln(1+\gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2. \tag{\#}$$

The gap in (#) appears because ECDQs generate uniform (and not Gaussian) noise, and because practical EC-ED pairs are not perfectly efficient.

Main result

Consider the Main Problem again.

Theorem

If $EH \neq 0$ at the optimum, and $\left[\sigma_y^2\right]_\Gamma = D^*$, then
$$\mathcal{R}_{D^*} \geqslant \frac{1}{2}\ln(1+\Gamma).$$
Moreover, there exists a linear source coding scheme using an ECDQ such that
$$\mathcal{R} < \frac{1}{2}\ln(1+\Gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$
while satisfying $\sigma_y^2 = D^* + \epsilon$ for any $\epsilon > 0$.

- We found a lower bound on $\mathcal{R}_D$, when linear source coding schemes are employed to control SISO LTI plants.
- Our bound is tight up to $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats per sample (i.e., ≈ 1.254 bits per sample).

Our approach is constructive and built upon the solution of a convex SNR optimization problem.

Corollary

Denote the minimal average data-rate that is compatible with MSS by $\mathcal{R}_{\text{MSS}}$. Then,
$$\sum_{i=1}^{n_p}\ln|p_i| \leqslant \mathcal{R}_{\text{MSS}} < \sum_{i=1}^{n_p}\ln|p_i| + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$

We achieve MSS at average data-rates that are at most $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats per sample away from the absolute lower bound established by [Nair & Evans, 2004].

Conclusions

- Using a class of coding schemes, we established lower and upper bounds on $\mathcal{R}_D$.
- Instrumental to our results was the solution of an SNR constrained optimal control problem.
- The latter problem is convex and hence amenable to efficient numerical implementation.
- Our results provide a bridge between control and information theories.