Georgia Robotics and InTelligent Systems Laboratory

http://gritslab.ece.gatech.edu

# Index-Free Multiagent Systems
## An Eulerian Approach

Peter Kingston and Magnus Egerstedt


## Abstract

Since the properties of systems comprising many homogeneous agents may be expected to be independent of how the agents happen to be indexed, it should be possible to formulate and solve multiagent control problems in an index-free way. In this paper we provide such an approach, based on an indicator distribution representation, which results in integro-differential dynamics that parallel and extend those obtained within the traditional indexed formulations. Conservation and stability properties are proven; a compatible geometric structure is constructed for the indexed representation; and a discrete analogue is presented which illustrates that for certain problems the Eulerian viewpoint results in very simple controllers.


## Lagrangian Representation

*"What state is each agent in?"*

### A Multiagent System

- $N$ agents with states $x_1, \cdots, x_N \in \mathbb{R}^n$
- Collected in joint state vector $x = (x_1, \cdots, x_N) \in \mathbb{R}^{nN}$

### Permutation Invariance

Equivalence relation:

$$x \sim y \iff \exists P \in \pi(N) \text{ s.t. } x = (P \otimes I)y$$

where $\pi(N)$ denotes the group of $N \times N$ permutation matrices. I.e., invariant under identity swaps:

$$(x_1, x_2, x_3) \sim (x_2, x_1, x_3)$$

## Eulerian Representation

*"How many agents are in each state?"*

### Indicator Distribution

$$m(x) = \Phi(x_1, \cdots, x_N)(x) = \sum_{i=1}^{N} \delta(x - x_i)$$

where $\delta$ is the Dirac delta distribution on $\mathbb{R}^n$, $\mathcal{T}(\mathbb{R}^n)$ denotes the space of tempered distributions on $\mathbb{R}^n$, and the map $\Phi : \mathbb{R}^{nN} \to \mathcal{T}(\mathbb{R}^n)$ creates $m$ from $x_1, \cdots, x_N$.

### Identical Controllers

In Lagrangian representation,

$$\dot{x}_i = v_i$$
$$v_i(t) = v(x_i(t), m(\cdot, t))$$

yields,

### Advection equation

$$\begin{aligned} \dot{m} &= -\operatorname{div}(mv) \\ &= -(\nabla m) \cdot v - m(\nabla \cdot v) \ . \end{aligned} \qquad (1)$$

## Linear Consensus

### Lagrangian setting

The joint state vector evolves according to

$$\dot{x}(t) = -L_w(\mathcal{G}(t))x(t)$$

where $L_w(\mathcal{G}(t))$ is the *graph Laplacian* for the interaction graph $\mathcal{G}(t)$.

### Eulerian setting

The indicator distribution has the dynamics (1) with

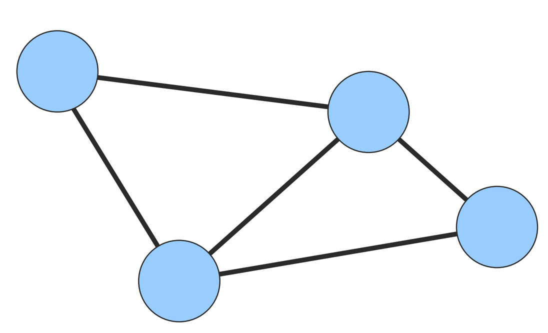
$$v = \int_{\zeta \in \mathbb{R}^n} m(\zeta) w(\zeta, x, m)(\zeta - x) d\zeta \ .$$

### Properties of Index-Free Linear Consensus

- **Theorem 1:** Center of Mass Conservation
- **Theorem 2:** Stability

## A Finite-State Analogue

### A Graph of Rooms



- State: For each room $i$, a number $m_i \in \mathbb{R}$ of agents currently in that room.

### Eulerian Dynamics

$$\left.\begin{aligned} m[k+1] &= m[k] + Du[k] \\ \text{s.t.} \quad m[k] &\geq 0 \end{aligned}\right\} \forall k \in \mathbb{N}$$

where $D$ is the incidence matrix associated with $\mathcal{G} = (V, E)$ and $u[k] \in \mathbb{R}^{|E|}$.

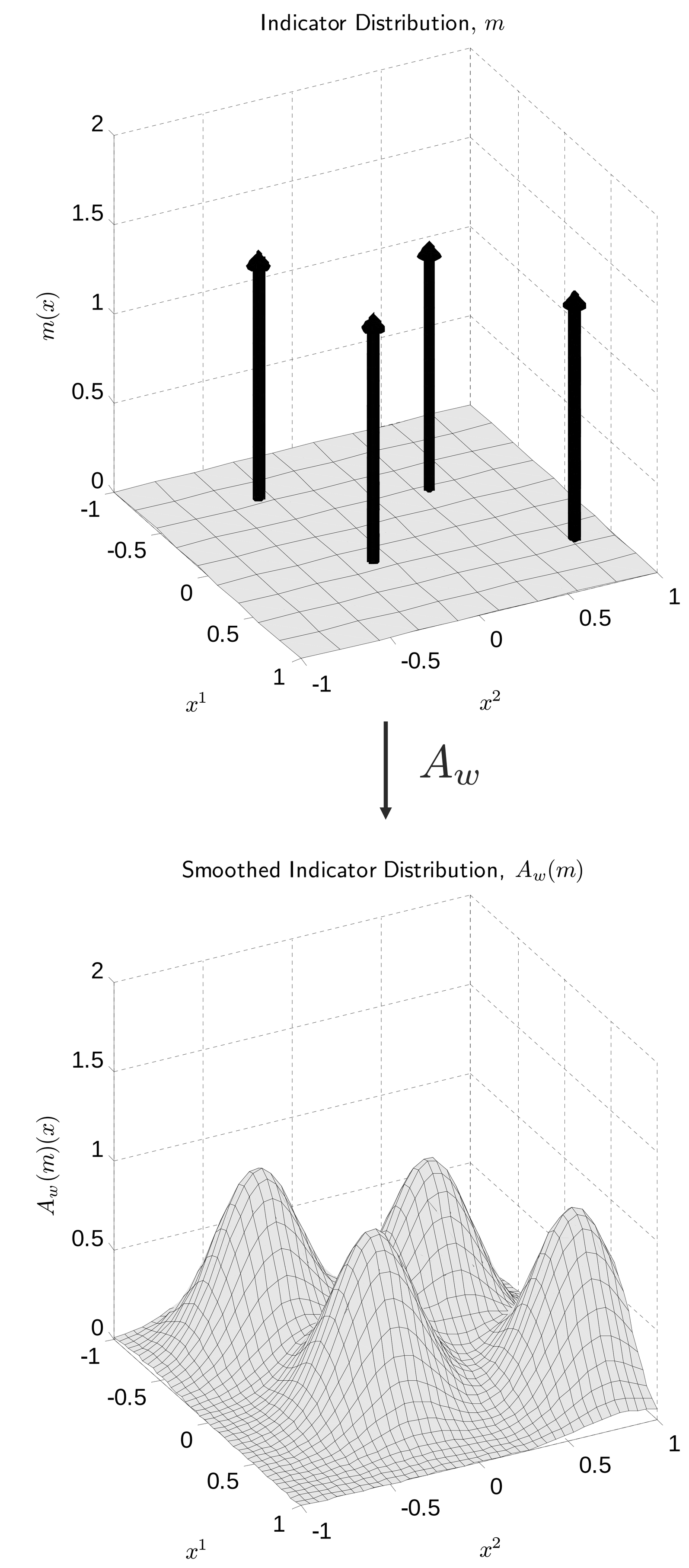
#### A Coverage Controller

$$u[k] = -\gamma D^T m[k] \ , \gamma > 0$$
$$\Downarrow$$
$$m[k+1] = [I - \gamma L(\mathcal{G})] m[k]$$

## Inner Product Space via Smoothing

*Imbue state space with geometric structure.*

### Smoothing




### Kernelization

*Compute directly with Lagrangian representation.*

The *kernel* $\kappa$ attached to the embedding $A_w \circ \Phi$ is the map

$$\kappa((x_1^1, \cdots, x_N^1), (x_1^2, \cdots, x_N^2)) \triangleq \left\langle A_w(\Phi(x_1^1, \cdots, x_N^1)), A_w(\Phi(x_1^2, \cdots, x_N^2)) \right\rangle_{L^2}$$

which, for Gaussian smoothing,

$$= \sqrt{\frac{\pi^n}{\det Q}} \sum_{i,j} \exp\left(-\frac{1}{2}(x_j^2 - x_i^1)^T Q (x_j^2 - x_i^1)\right) .$$

## Contributions

- Permutation-invariant representation.
- Integro-differential model.
- Properties of index-free consensus.
- Inner product structure and kernel.
- Explored Eulerian-Lagrangian / Coverage-consensus duality.

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