# Improved stability analysis of networked control systems under asynchronous sampling and input delay

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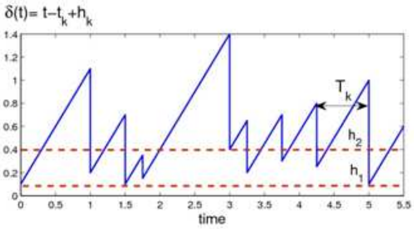
This article presents a novel approach to assess the stability of linear systems with delayed and sampled-data inputs. It proposes an extension of existing results on the stability of sampled-data systems to the case where a delay is introduced in the control loop. The method is based on a continuous-time modelling of the systems together with the discrete-time Lyapunov theorem, which provides easy tractable sufficient conditions for asymptotic stability. Those conditions cope the problem of stability under asynchronous samplings and time-varying delays. The period and delay-dependent conditions are expressed using computable linear matrix inequalities.

## Contribution

- A novel approach to assess the stability of linear systems with delayed and sampled-data inputs;
- Continuous-time modeling of the systems together with the discrete-time Lyapunov theorem;
- Easy tractable sufficient conditions are provided for asymptotic stability;
- The two types of delays are considered separately;
- A larger upper-bound of the allowable sampling period is given.

## Influence of delay, samplings




Examples of a delay generated by a transmission delay $h_k$ bounded by $h_1$ and $h_2$ and an asynchronous sampling of periods $T_k$

## Conclusion

- A novel analysis of NCS under asynchronous sampling and input delay is provided in this article;
- The examples show the efficiency of the method;
- The reduction of the conservatism compared to other results from the literature.

## References

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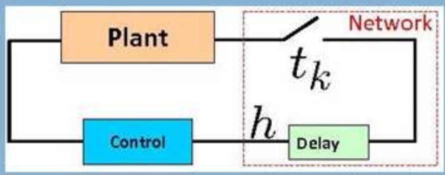
## Problem formulation




Control loop of the networked control systems under transmission and sampling delays

Consider the linear system with a sampled and delayed input as shown in the left figure:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad (1)$$

We are looking for a piecewise-constant static state-feedback control law:

$$u(t) = Kx(s_k), \quad s_k + h(s_k) \le t < s_{k+1} + h(s_{k+1})$$

where $0 = s_0 < s_1 < \dots < s_k < \dots$ represent the sampling instants.

## Main results

**Theorem 1** *Let $V : \mathbb{K}^{h_2} \to \mathbb{R}^+$ be a functional for which there exist real numbers $0 < \mu_1 < \mu_2$ and $p > 0$ such that*

$$\forall (x_t) \in \mathbb{K}, \quad \mu_1 |x_t(0)|^p \le V(x_t) \le \mu_2 |x_t|^p. \quad (1)$$

*The two following statements are equivalent.*
*(1) $\forall k \ge 0, \quad \Delta V(k) = V(x_{t_{k+1}}) - V(x_{t_k}) < 0$;*
*(2) There exists a continuous functional $\mathcal{V} : \mathbb{R} \times \mathbb{K}^{h_2}_T \to \mathbb{R}$, differentiable over all intervals of the form $[t_k\ t_{k+1}[$ which satisfies*

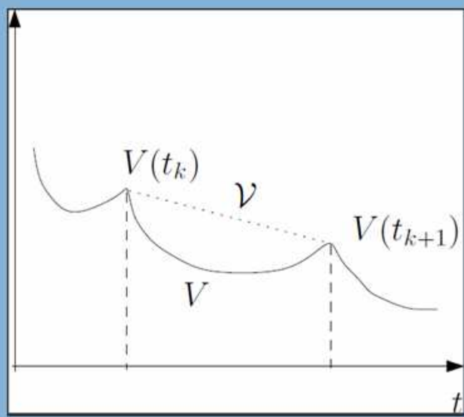
$$\forall k \ge 0, \quad \mathcal{V}(T_k, \chi_k^{h_2}) = \mathcal{V}(0, \chi_k^{h_2}). \quad (2)$$

*and such that, for all $k > 0$ and for all $t$ in $[t_k\ t_{k+1}]$, the following inequality holds*

$$\mathcal{W}(\tau(t), \chi_k^{h_2}) < 0, \quad (3)$$

*where $\mathcal{W}(\tau(t), \chi_k^{h_2}) = \frac{d}{dt}\left\{[V(x_t) + \mathcal{V}(\tau(t), \chi_k \chi_k^{h_2})]\right\}$. Moreover, if one of these two statements is satisfied, the solutions of system with the given control law are asymptotically stable.*

## Sketch of the proof




## Stability criteria

**Theorem 2** For given $0 \le T_1 < T_2$ and $0 \le h_1 < h_2$, consider an asynchronous sampling and a time-varying delay $\boldsymbol{h}$ which satisfy (1) and (2). Assume that there exist, for $p = 1,2,3$ and for $l = 1,2,3,4$, the matrices $Q_p$ and $R_l \in \mathbb{S}^n_+$, $P$ and $U \in \mathbb{S}^{2n}_+$, $S_1$ and $X \in \mathbb{S}^{2n}$ and three matrices $S_2 \in \mathbb{R}^{2n\times 2n}$, $Y_1$, $Y_2 \in \mathbb{R}^{2n\times n}$, $Y_3 \in \mathbb{R}^{5n\times 2n}$, that satisfy for $i = 1,2$ and $j = 1,2$

$$\begin{bmatrix} \Pi_1(h_j) + \bar{T}_i(N_2^T X N_2 + \Pi_2) & h_{21}N_1Y_j \\ * & -h_{21}\Pi_{3j} \end{bmatrix} < 0, \qquad (1)$$

$$\begin{bmatrix} \Pi_1(h_j) - \bar{T}_i N_2^T X N_2 & \bar{T}_i N_3 Y_3 & h_{21}N_1Y_j \\ * & -\bar{T}_i U & 0 \\ * & * & -h_{21}\Pi_{3j} \end{bmatrix} < 0, \qquad (1)$$

$$Q_{23} = Q_2 - Q_3 > 0 \qquad (2)$$

where $\bar{T}_1 = \max(0, T_1 - h_{21})$, $\bar{T}_2 = T_2 + h_{21}$. The system is thus asymptotically stable for the sampling period $T$ and the time-varying input delay $h$.

## Examples

**Example 1** Consider system with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_d = BK = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \begin{bmatrix} -3.75 \\ -11.5 \end{bmatrix}^T.$$

**Example 2** Consider system with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

**Example 3** Consider system with

$$\dot{x}(t) = -x(t_k - h).$$

Concerning the time-varying transmission delay, we consider $h_1 = 10^{-4}$, $\epsilon_2 = 1$ and $\epsilon_1 = -0.2$ and $-1$. The results delivered Theorem 2 for examples 1, 2 and 3 are summarized in Table. They show the influence of the delay variation $\epsilon_1$.

| Ex.1 | | | | | | | |
|---|---|---|---|---|---|---|---|
| $h_2$ | $10^{-3}$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.061 |
| [Naghshtabrizi et al. 2010] | 1.111 | 0.714 | 0.469 | 0.269 | 0.069 | - | - |
| [Millán et al. 2009] | 1.042 | 0.843 | 0.643 | 0.443 | 0.243 | 0.043 | - |
| [Liu and Fridman 2009b] | 1.638 | 1.063 | 0.786 | 0.541 | 0.301 | 0.054 | - |
| Th. 2, $\epsilon_1 = -0.2$ | 1.671 | 1.307 | 1.034 | 0.76 | 0.462 | 0.121 | $10^{-3}$ |
| Th. 2, $\epsilon_1 = -1$ | 1.640 | 1.049 | 0.798 | 0.564 | 0.322 | 0.059 | - |
| Ex.2 | | | | | | | |
| $h_2$ | $10^{-3}$ | 0.5 | 1 | 1.5 | 2 | 3 | 3.307 |
| [Naghshtabrizi et al. 2010] | 1.278 | 0.499 | - | - | - | - | - |
| [Millán et al. 2009] | 1.867 | 1.368 | 0.868 | - | - | - | - |
| [Liu and Fridman 2009b] | 1.970 | 1.368 | 0.868 | 0.443 | 0.212 | 0.038 | - |
| Th. 2, $\epsilon_1 = -0.2$ | 2.511 | 1.878 | 1.666 | 1.424 | 1.167 | 0.512 | $10^{-3}$ |
| Th. 2, $\epsilon_1 = -1$ | 2.444 | 1.362 | 1.130 | 0.856 | 0.484 | - | - |
| Ex.3 | | | | | | | |
| $h_2$ | $10^{-3}$ | 0.1 | 0.3 | 0.6 | 0.9 | 1.2 | 1.346 |
| [Naghshtabrizi et al. 2010] | 1.278 | 1.064 | 0.704 | 0.399 | 0.099 | - | - |
| [Millán et al. 2009] | 1.338 | 1.239 | 1.039 | 0.739 | 0.439 | 0.139 | - |
| [Liu and Fridman 2009b] | 1.945 | 1.548 | 1.230 | 0.864 | 0.522 | 0.168 | - |
| Th. 2, $\epsilon_1 = -0.2$ | 1.945 | 1.783 | 1.560 | 1.208 | 0.814 | 0.335 | $10^{-3}$ |
| Th. 2, $\epsilon_1 = -1$ | 1.914 | 1.473 | 1.209 | 0.890 | 0.551 | 0.179 | - |

Maximal sampling period $T$ for time-varying transmission delays(Theorem 2) for examples 1, 2 and 3.

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