On the Structural Controllability of Networks of Linear Systems

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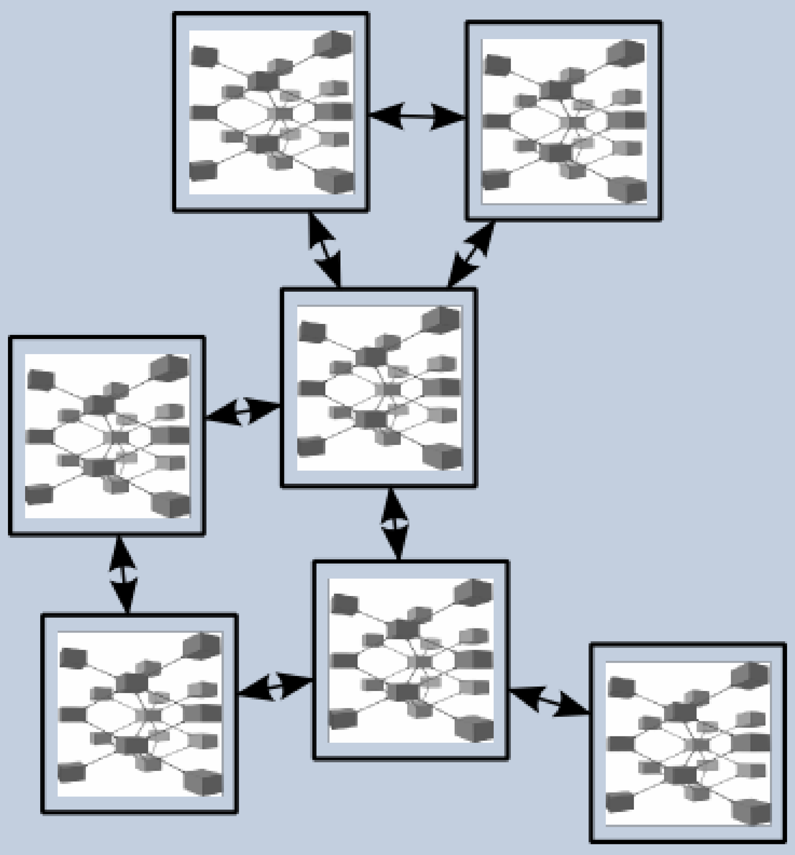
Motivation and Problem (1)

Just as the notion of controllability has underpinned the analysis of dynamical systems for many decades, we now look to find criteria to determine the controllability of dynamical networks. We thus wish to determine the joint (or global) controllability of the dynamical network composed of interconnected dynamical systems where each individual dynamic system has the structured systems model:

$$\dot{\mathbf{x}}_\mathbf{k} = \bar{A}_k \mathbf{x}_\mathbf{k} + \bar{B}_k \mathbf{u}_\mathbf{k} + \sum_j \bar{L}_{kj} \mathbf{x}_\mathbf{j} \tag{1}$$



The network model in Eqn. (1) can be thought of as the network of networks paradigm.

Structured Systems (2)

A *structured linear system* is given by:

$$\dot{\mathbf{x}} = \bar{A}\mathbf{x} + \bar{B}\mathbf{u} \tag{2}$$

where the interconnection matrices $\bar{A}$ and $\bar{B}$ are obtained from the matrices A and B of a linear system as:

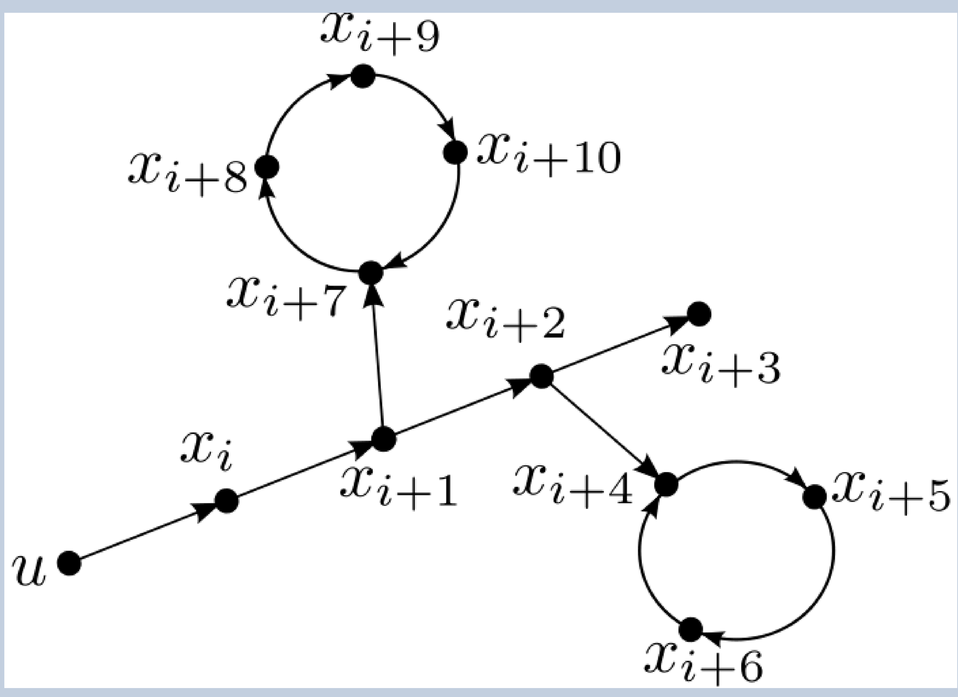
$$\bar{a}_{ij} = \begin{cases} 1 & a_{ij} \neq 0 \\ 0 & a_{ij} = 0 \end{cases} \quad \bar{b}_{ij} = \begin{cases} 1 & b_{ij} \neq 0 \\ 0 & b_{ij} = 0 \end{cases}$$

The binary interconnection structure can be represented as a directed graph, *a digraph*, denoted by $D_{(A,B)}$.
Given a structured linear system written in the form:

$$\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{1} \\ \dot{\mathbf{x}}_\mathbf{2} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathbf{1} \\ \mathbf{x}_\mathbf{2} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \mathbf{u} \tag{3}$$

then the state variables $\mathbf{x}_\mathbf{1}$ are *structurally state variable controllable* if there generically exists an input $\mathbf{u}$ that will drive the state variables $\mathbf{x}_\mathbf{1}$ from an arbitrary initial state to an arbitrary final state.

Cacti (3)



A cactus with a single stem and two buds. The node x_{i+3} is a terminal stem vertex.

Structural Controllability and Cactus Covers (4)

Theorem 1. *The state variables* $\mathbf{x}_\mathbf{1}$ *of Eqn.* (3) *are structurally state variable controllable iff the state variables* $\mathbf{x}_\mathbf{1}$ *have a cactus cover (a disjoint union of cacti) contained in the system digraph* $D_{(A,B)}$

Theorem 2. *In the linear structured system given by Eqn.* (2), *with generic controllable dimension* r, *and with digraph* $D_{(A,B)}$ *there exists* r *state variable vertices with a cactus cover. These vertices are structurally state variable controllable. Furthermore, there is no set of* $r+1$ *state variable vertices for which a cactus cover exists or which are structurally state variable controllable.*

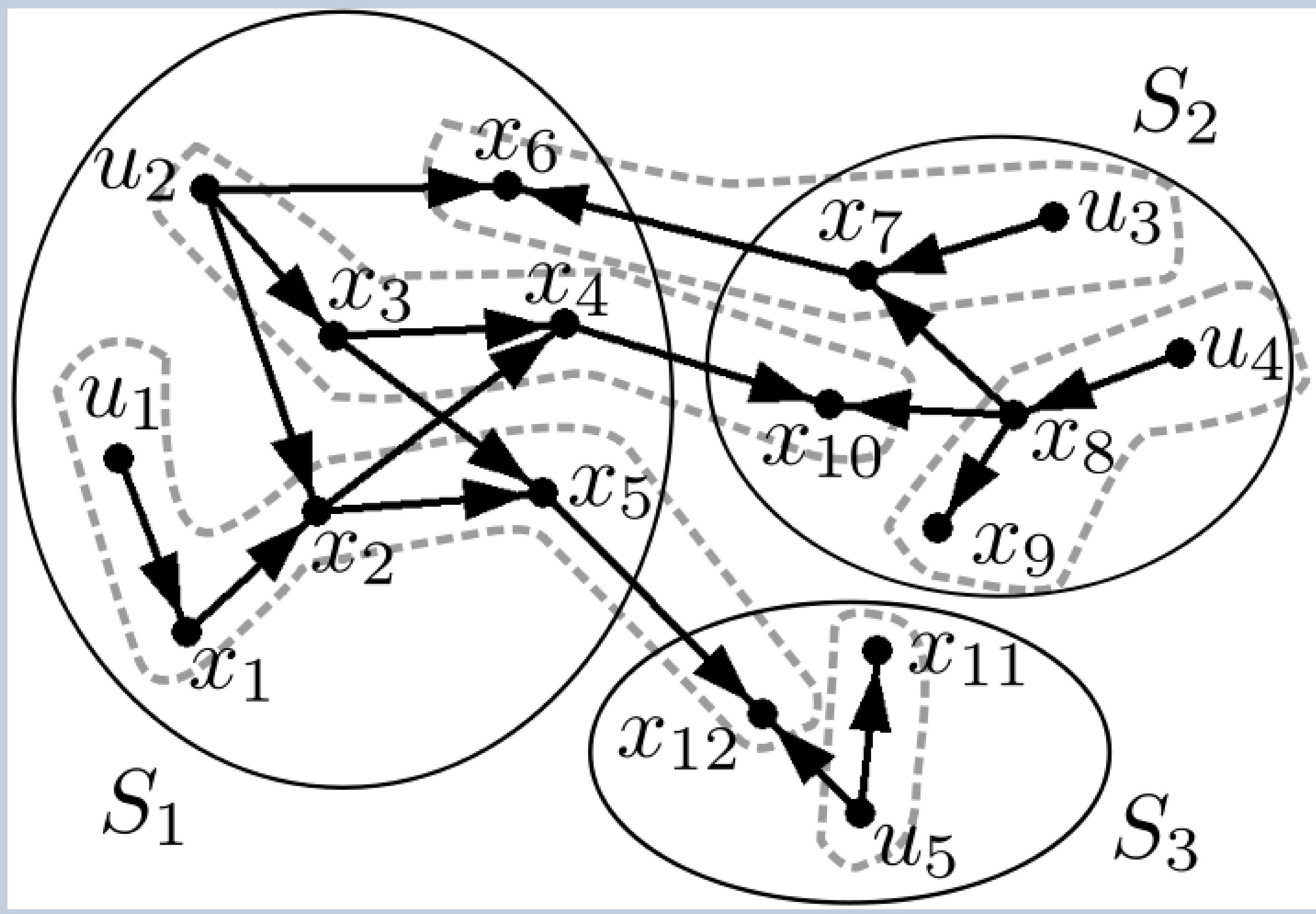
Main Result - Network Structural Controllability (5)

Theorem 3. *The structured dynamical network of Eqn.* (1) *is jointly structurally controllable (i.e. all state variables are structurally state variable controllable) iff the following two conditions are satisfied*

(C1) Each individual system is virtually structurally controllable with respect to some local and virtual controllers (i.e. incoming network interconnections), that is the pair $(\bar{A}_k, [\bar{B}_k, \bar{L}_{k1}, \cdots, \bar{L}_{k\kappa}])$ *is structurally controllable and;*

(C2) Every virtual controller of (C1) is connected to an unique terminal stem vertex.

Example (6)



In this example none of the three systems are structurally controllable prior to interconnection but using the results of Thm. 3 it can be seen that the network as a whole is structurally controllable after interconnection.

Conclusion and Future Work (7)

The main result provides a criterion for determining the structural controllability of dynamical networks using only local information about each system in the network and the network interconnection structure. In proving these results the authors have provided some new notions of controllability, namely the notion of state variable controllability.

It will be beneficial to extend these results to structured hybrid and nonlinear systems, to investigate the structural stabilisability of networks of linear systems and to hypothesise on what other properties of linear and structured systems may be defined for an initial choice of state variables, instead of being a system wide property.