Optimal Redundant Transmission for State Estimation with Packet Drops

Girish N. Nair[1], Alexandre R. Mesquita[2], João P. Hespanha[2]

[1]Dept. Electrical & Electronic Engineering, U. Melbourne, Australia, gnair@unimelb.edu.au

[2]Dept. Electrical & Computer Engineering, U. California Santa Barbara, USA

Problem Statement

Consider a networked dynamical system in which measurements are transmitted over a packet-dropping channel to a state estimator.

Suppose the transmitter can send multiple packets per sample. An obvious wayto improve the chances of successful reception is to send duplicate packets for each sample.

However, this also increases the communication rate (average #packets/sample).

If the transmitter gets acknowledgements (ACK's) from the receiver, how can it dynamically choose the #packets/sample to transmit, in order to achieve an 'optimal' trade-off?

Formulation

Plant : $X_{t+1} = AX_t + W_t \in \mathbb{R}^n$, fully observed.

W Mean zero white noise $\perp X_0$, cov[W]= Σ_W.

Channel : Drops packets independently,
P[Packet drop] =: p, i.i.d.

Say $N_t \geq 0$ identical packets sent during $[t, t+1)$.

$\Rightarrow \mathrm{P}\left[\text{Sending fails during } [t,t+1)\right] = p^{N_t}$.

$$\text{Estimator}: \hat{X}_{t+1} = \begin{cases} AX_t & \text{if sending succeeded} \\ A\hat{X}_t & \text{if sending failed.} \end{cases}$$

A binary symbol ACK$_t$, indicating whether sending succeeded or failed, is then conveyed back to the transmitter before time $t+1$.

Optimal 'Simple' Policy [Mesquita et. al., ACC09]

$$J := \overbrace{\varlimsup_{k\to\infty} \frac{1}{k} \mathrm{E}\left[\sum_{t=0}^{k-1} E_t Q E_t^{\mathrm{T}}\right]}^{\text{Estimation Cost}} + \lambda \cdot \overbrace{\varlimsup_{k\to\infty} \frac{1}{k} \mathrm{E}\left[\sum_{t=0}^{k-1} N_t\right]}^{\text{Communication Rate}},$$

estimation error $E_t := X_t - \hat{X}_t$, weight $\lambda > 0$

Suppose that N_t can depend only on received ACK's, not directly on X_t or E_t.

$\Rightarrow$ Partially observed MDP (assuming W i.i.d.), with integer action variable $N_t \in \{0,1,2,\ldots\}$.

Under certain technical conditions, $\exists$ a stationary optimal policy for N_t, which furthermore depends only on current no. L_t of sending failures since last success.

Cost via Stationary Distribution

Under a policy $N_t \equiv \nu(L_t)$, (L_t) is a homogeneous, irreducible Markov chain with stationary dist.

$$\mu(m) := \lim_{t\to\infty} \mathrm{P}\left[L_t = m\right] = \frac{p^{\sum_{k=0}^{m-1}\nu(k)}}{\sum_{i\geq 0} p^{\sum_{k=0}^{i-1}\nu(k)}}$$

$$\Rightarrow J \equiv J(\nu) = \frac{\sum_{i\geq 0}\left(g(i) + \lambda\nu(i)\right) p^{\sum_{k=0}^{i-1}\nu(k)}}{\sum_{i\geq 0} p^{\sum_{k=0}^{i-1}\nu(k)}} \equiv \frac{N(\nu)}{D(\nu)},$$

where $g(i) := \sum_{k=0}^{i} \mathrm{tr}\left(QA^k \Sigma_W A^{k\mathrm{T}}\right)$.

Continuous & differentiable w.r.t. **real** $\nu(0), \nu(1)\ldots$

$\Rightarrow$ What if we relax integer constraint & differentiate?

Recursion for Relaxed Minimum

Let $\mathrm{V} := \left\{\nu \in \mathbb{R}_{\geq 0}^{\infty} : N(\nu), D(\nu) < \infty\right\}$

At any global minimum $\nu^* \in \mathrm{V}$ with cost J^*,

$$\frac{\partial N(\nu)}{\partial \nu(m)} - J\left.\frac{\partial N(\nu)}{\partial \nu(m)}\right|_{\nu=\nu^*} \begin{cases} = 0 & \text{if } \nu^*(m) > 0 \\ \geq 0 & \text{if } \nu^*(m) = 0 \end{cases}.$$

After some manipulations, we find that ν^* must be uniquely given by a coupled recursion

$$\nu_\alpha(m) := \left[\kappa_\alpha(m-1) + \frac{1}{\ln p} + \frac{\alpha - g(m)}{\lambda}\right]^+,$$

$$\kappa_\alpha(m) := \begin{cases} \dfrac{-p^{-\nu_\alpha(m)}}{\ln p} & \text{if } \nu_\alpha(m) > 0 \\ \kappa_\alpha(m-1) + \dfrac{\alpha - g(m)}{\lambda} & \text{if } \nu_\alpha(m) = 0 \end{cases},$$

with parameter $\alpha = J^* \in \mathbb{R}_+$ and $\kappa_\alpha(-1) := 0$.

$\Rightarrow$ Instead of searching over infinite-dim. space V, do a one-dim. search over policies of the form ν_α.

Main Result (Unbounded # Packets)

Let $m_\alpha := \min\left\{m \in \mathbb{Z}_{\geq 0} : g(m) > \alpha\right\} < \infty$,

$\gamma_\alpha(m) := 1 - \left(g(m+1) - \alpha\right)(\ln p)/\lambda$,

$$\zeta_\alpha(m) := \frac{-1}{\ln p} \ln\left(\gamma_\alpha(m) + \ln\left(\gamma_\alpha(m+1) + \ln(\cdots)\right)\right)$$

Then $\forall \alpha > 0$, the following statements are equivalent.

(A) $\nu_\alpha(m) \geq \zeta_\alpha(m)$ when $m = m_\alpha$

(B) $\nu_\alpha \in \mathrm{V}$

(C) $\forall\ m \geq m_\alpha,\ \nu_\alpha(m) \geq \zeta_\alpha(m) \to \infty$.

(D) $J(\nu_\alpha) = \alpha$

Corollaries :

$$\min_{\nu\in\mathrm{V}} J(\nu) = \min\{\alpha > 0 : \nu_\alpha(m_\alpha) \geq \zeta_\alpha(m_\alpha)\}$$
$$= \min\{\alpha > 0 : J_\alpha(\nu_\alpha) \leq \alpha\}$$

I.e., reduce α in small steps until (A) or (D) violated.

Recursive Solution (Bounded Case)

In practice, # packets/sample is bounded.

If we impose $\nu(m) \leq V$, then $\nu^* = \eta_\alpha|_{\alpha=J^*}$, where

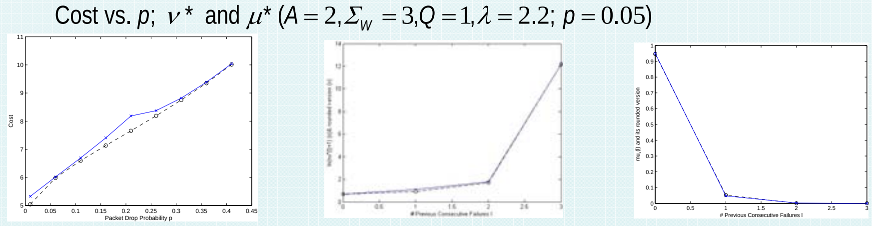
$$\eta_\alpha(m) := \min\left(\left[\phi_\alpha(m-1) + \frac{1}{\ln p} + \frac{\alpha - g(m)}{\lambda}\right]^+, V\right),$$

$$\phi_\alpha(m) := \begin{cases} \dfrac{1}{p^V}\left(\phi_\alpha(m-1) + \dfrac{\alpha - g(m)}{\lambda} - V\right), & \text{if } \eta_\alpha(m) = V \\ \dfrac{-p^{-\eta_\alpha(m)}}{\ln p}, & \text{if } 0 < \eta_\alpha(m) < V \\ \phi_\alpha(m-1) + \dfrac{\alpha - g(m)}{\lambda}, & \text{if } \eta_\alpha(m) = 0 \end{cases}.$$

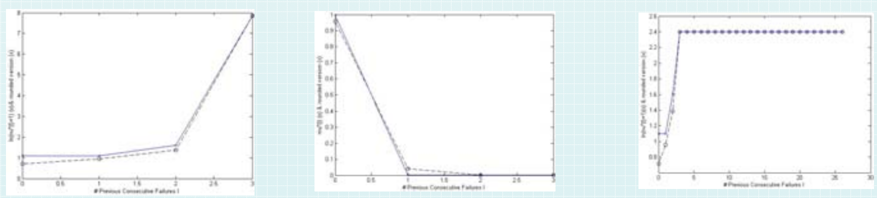
Analogous properties hold in this case.

Numerical Results

Cost vs. p; ν^* and μ^* ($A = 2, \Sigma_W = 3, Q = 1, \lambda = 2.2$; $p = 0.05$)



Plots for $\left(A = \begin{bmatrix} 2 & 1 \\ 0 & .5 \end{bmatrix}, \Sigma_W = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \lambda = 2.2, p = 0.05\right)$



Unbounded. J=9.04

V=10. J=9.04