A Sub-Optimal Approach to Design Distributed Controllers Realizable over Arbitrary Networks

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Abstract

- In this paper, we consider the problem of designing distributed output-feedback controllers that achieve H_2 and H_∞ performance objectives for a particular class of interconnected systems that are formed by interactions over an arbitrary directed communication network.
- The problem is formulated and analyzed in terms of the state-space parameters for the sake of network realizability.
- For a particular class of discrete-time linear time-invariant interconnected systems that are characterized by a structural property of their state-space matrices, we provide sufficiency conditions for designing stabilizing distributed controllers which can use the available network along with the subsystems of the interconnected system.
- A parametrization is considered for the output-feedback linear distributed controllers that allows minimization of closed-loop H_2 and H_∞ norms to be expressed as *semi-definite programs* (SDPs).
- If a solution exists for the SDPs, then the solution allows us to synthesize the corresponding stabilizing distributed controller realizable over the given network.
- In this design process, a trade-off is made between optimality and network realizability of the distributed controller.

Interconnected systems

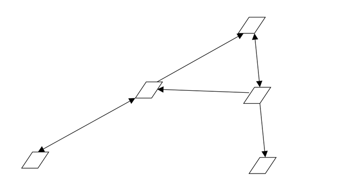

Fig 1: Example of an interconnected system

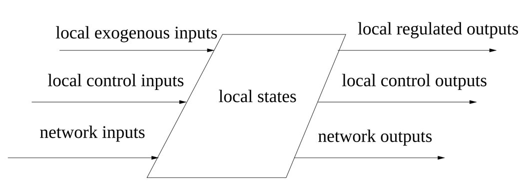

Fig 2: Description of a subsystem in the network

- A group of subsystems interacting over a given communication network is referred to as a *Networked* or *Interconnected system*.
- For an interconnected system to be described over a network, it needs to satisfy certain constraints.
- When the interconnected system is described by its transfer function, the constraints required for the system to be compatible with the network are given by the corresponding delay and sparsity constraints (refer to [Rot05]).
- But these constraints are not sufficient for the system to be realizable over the given network.

Network realizability

- If the transfer function of a given interconnected system can be expressed in state-space form as separate subsystems interacting over a given network, then the system is said to be *network realizable*.
- In some special cases, for example when the communication network corresponds to an acyclic graph, one can realize the transfer function of the interconnected system over the given network (refer to [SL09]).
- For general interconnected systems, even if the transfer function satisfies the required delay and sparsity constraints, it is not known how to realize the system over the network.
- In this paper, we design interconnected systems compatible with the network interconnection by imposing constraints on their state-space parameters.
- We show that, realizability of an interconnected system over a given network (with the given state, input and output partitions), corresponds to its state-space matrices being structured according to the graph (i.e. sparsity constraints have to be imposed on the state-space matrices based on the adjacency matrix of the graph).

Example

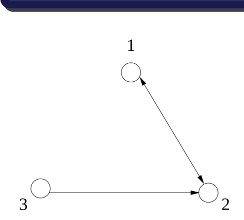

Fig 3: A three node system

Consider a simple three node interconnected system with a communication network described by a digraph as shown in Fig. 3 with adjacency matrix $\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$. This means that the subsystems at node 1 and 2 can exchange messages while the one at node 3 can send messages to the subsystem at node 2.

Let the dynamics of the three subsystems be given by the following state-space equations

$$\begin{bmatrix}x_1(k+1)\\y_1(k)\\\eta_{12}(k)\end{bmatrix}=\begin{bmatrix}0.5&1&1\\1&0&0\\0.8&0&0\end{bmatrix}\begin{bmatrix}x_1(k)\\u_1(k)\\\eta_{21}(k)\end{bmatrix},\quad\begin{bmatrix}x_2(k+1)\\y_2(k)\\\eta_{21}(k)\end{bmatrix}=\begin{bmatrix}1.5&2&1&1\\1&0&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}x_2(k)\\u_2(k)\\\eta_{12}(k)\\\eta_{32}(k)\end{bmatrix},$$

$$\begin{bmatrix}x_1(k+1)\\y_1(k)\\\eta_{32}(k)\end{bmatrix}=\begin{bmatrix}0.7&1\\2&0\\0.7&0\end{bmatrix}\begin{bmatrix}x_3(k)\\u_3(k)\end{bmatrix}.$$

where $\eta_{ij}(k)$ denote the messages passed over the network from node i to node j at the k^{th} instant. Combining the above dynamics, the resultant dynamics for the three node interconnected system is given by

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\x_3(k+1)\end{bmatrix}=\begin{bmatrix}0.5&1&0\\0.8&1.5&0.7\\0&0&0.7\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}+\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}u_1(k)\\u_2(k)\\u_3(k)\end{bmatrix},\quad\begin{bmatrix}y_1(k)\\y_2(k)\\y_3(k)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}.$$

- Following the simple algebra to obtain the dynamics of the interconnected system, one can observe that the sparsity of the A matrix matches with the sparsity of the adjacency matrix.
- This observation is presented in the paper for a general setting where the dimensions of the local state, control input and output vectors of each subsystem can be different.
- The state-space representation of a given interconnected system is compatible with the given network interconnection if and only if the sparsity patterns of the state-space matrices comply with the adjacency matrix of the given graph.

The transfer function matrix corresponding to this interconnected system is given by

$$\begin{bmatrix}\frac{2}{z^2-2z-0.05}&\frac{2}{z^2-2z-0.05}&\frac{0.7}{z^3-2.7z^2+1.35z+0.035}\\\frac{0.8}{z^2-2z-0.05}&\frac{2z-1}{z^2-2z-0.05}&\frac{0.7z-0.35}{z^3-2.7z^2+1.35z+0.035}\\0&0&\frac{2}{z-0.7}\end{bmatrix}.$$

- It can be observed that the above transfer function satisfies the delay and sparsity constraints that were mentioned earlier.
- But given the transfer function alone, it is non-trivial to obtain a realization that satisfies the sparsity constraints on the corresponding state-space matrices.
- Also note that the network realization should not introduce more unstable poles than that of the transfer function.

This example is given to emphasize the fact that network realization of even simple interconnected systems can be non-trivial. For this reason, even though frequency domain approaches exist for solving optimal distributed control problems, it should be noted that the solution may not be realized over the given network.

Main result

- In this paper, we consider interconnected systems with state-space representation given by

$$\begin{bmatrix}x(k+1)\\z(k)\\y(k)\end{bmatrix}=\begin{bmatrix}A&B_w&B_u\\C_z&D_{zw}&D_{zu}\\C_y&D_{yw}&0\end{bmatrix}\begin{bmatrix}x(k)\\w(k)\\u(k)\end{bmatrix}\qquad(1)$$

  where A, B_w, C_z and D_{zw} have a sparsity structure based on the network interconnection, while B_u, C_y, D_{zu} and D_{yw} have a block diagonal structure.
- To overcome the problems of network realization, we propose a distributed controller design strategy where sparsity constraints are imposed on the state-space matrices of the controller.
- The following theorem provides a sufficiency condition for the existence of a stabilizing distributed controller that can also be realized over the given network while minimizing the H_2 norm of the closed-loop transfer function.
- This result is an application of the controller design approaches discussed in [SGC97] for a distributed setting. A similar result can be obtained in the case of H_∞ norm too.

Theorem

Given an interconnected system P with state-space structure defined by (1). If there exist matrices X, Y that are symmetric and block diagonal; Q, L, F, R that have sparsity structures based on the network; and a symmetric matrix W, where the dimensions of the constituent matrices of the block-diagonal and structured matrices are appropriately assigned, such that

$$0<\text{trace}(W)<\mu,\qquad(2)$$

$$\begin{bmatrix}W&C_z+D_{zu}RC_y&C_zX+D_{zu}L&D_{zw}+D_{zu}RD_{yw}\\(\cdot)'&Y&I&0\\(\cdot)'&(\cdot)'&X&0\\(\cdot)'&(\cdot)'&(\cdot)'&I\end{bmatrix}>0,\qquad(3)$$

$$\begin{bmatrix}Y&I&YA+FC_y&Q&YB_w+FD_{yw}\\(\cdot)'&X&A+B_uRC_y&AX+B_uL&B_w+B_uRD_{yw}\\(\cdot)'&(\cdot)'&Y&I&0\\(\cdot)'&(\cdot)'&(\cdot)'&X&0\\(\cdot)'&(\cdot)'&(\cdot)'&(\cdot)'&I\end{bmatrix}>0,\qquad(4)$$

then there exists a stabilizing controller K that is realizable on the given network, such that
$\|\mathrm{F}_l(P,K)\|_2^2<\mu$.

Remarks

- The main idea of the paper is to point out the importance of network realizability and the hidden difficulties in a frequency domain approach for designing distributed controllers over a network.
- In this paper, we considered a simple class of plants to make the analysis easier and yet convey the importance of realizability during controller design and synthesis.
- The sufficiency condition given in this paper ensures that if the semi-definite program has a solution, then there exists a stabilizing distributed controller that can be realizable over the given network.

References

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