NANYANG TECHNOLOGICAL UNIVERSITY

School of Electrical & Electronic Engineering

Average Consensus with Limited Data Rate and Switching Topologies

Tao Li and Lihua Xie

Abstract

This paper is concerned with discrete-time average-consensus with limited data-rate and switching communication topologies. We design a distributed encoding-decoding scheme based on difference quantization with dynamic scaling and a control protocol based on a symmetric compensation method. We develop an adaptive scheme to select the numbers of quantization levels. The number of quantization levels of each quantizer is tuned on-line according to whether the associated communication channel is active or not at the last step. We prove that if the network is jointly connected, then under the protocol designed, average-consensus can be achieved without steady-state error, and the convergence rate is quantified. Especially, if the duration of link failures of communication channels is bounded, then the control gain and the scaling function can be selected properly such that 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

Motivation

It is well known that in multi-agent networks the communication topology is often time-varying due to many reasons such as link failures or change of environment.

- ✓ Passive switching: link failures.
- ✓ Active switching: Network switches among different modes according to high level commands or for performance optimization.
- ✓ Most of the relevant literature on distributed consensus with quantized communication assume time-invariant communication topologies.

Distributed consensus with quantized communication and time-varying topologies is meaningful from both theoretic and engineering points of view.

Problem formulation

Dynamic model of agents: $x_i(t+1)=x_i(t)+u_i(t),\ t=0,1,\ldots,i=1,2,\ldots,N$

Network topology: a sequence of undirected graphs

$G(t)=\{V,A(t)\},t=1,2\ldots,$ $A(t)$ is the adjacency matrix of $G(t)$

Denote $N_i(t)=\{j\in V\mid a_{ij}(t)=1\},\ N_i=\bigcap_{t=1}^{\infty}\bigcup_{k=t}^{\infty}N_i(k)$

Objective: To achieve exact asymptotic average-consensus

Protocol design

For agent i and agent j, where $j\in N_i$, the encoder associated with i for channel (i,j) is designed as

$$\begin{cases}\xi_{ij}(0)=0,\\ \xi_{ij}(t)=g(t-1)a_{ij}(t)s_{ij}(t)+\xi_{ij}(t-1),\\ s_{ij}(t)=q_t^{ij}\left(\dfrac{x_i(t)-\xi_{ij}(t-1)}{g(t-1)}\right),\ t=1,2,\ldots,\end{cases}\qquad(1)$$

The decoder associated with agent j for channel (i, j) is given by

$$\begin{cases}\hat{x}_{ij}(0)=0,\\ \hat{x}_{ij}(t)=g(t-1)a_{ij}(t)s_{ij}(t)+\hat{x}_{ij}(t-1),\ t=1,2,\ldots,\end{cases}\qquad(2)$$

Here, $q_t^{ij}(\bullet)$ is the quantizer, whose number of quantization levels is $K_{ij}(t)$. $g(t)$ is the scaling function. We propose a distributed protocol as

$$u_i(t)=h\sum_{j\in N_i(t)}a_{ij}(t)(\hat{x}_{ij}(t)-\xi_{ij}(t)),\ t=0,1,\ldots,\quad i=1,2\ldots,N,\qquad(3)$$

where $h>0$ is the control gain.

Control parameters:

- ✓ Control gain: h
- ✓ Number of quantization levels: $K_{ij}(t),\ t=1,2,\ldots,i=1,2,\ldots,N,j\in N_i$
- ✓ Scaling function: $g(t)$

Main assumptions

A1) $N_i=\bigcup_{t=1}^{\infty}N_i(t),\ i=1,2,\ldots,N$

A2) There is a constant $d^*>0$ such that $\max_{1\le i\le N}\sup_{t\ge 0}d_i(t)\le d^*$

A3) $\max_i|x_i(0)|\le C_x,\max_{ij}|x_i(0)-x_j(0)|\le C_\delta$, where C_x and C_δ are known constants

A4) There exist an integer $T>0$ and a real constant $\lambda>0$ such that

$$\inf_{m\ge 0}\lambda_2\left(\sum_{k=mT+1}^{(m+1)T}L(k)\right)\ge\lambda(T+1)$$

Remark: Assumption A4) means the network flow is jointly-connected with the average algebraic connectivity being uniformly bounded below from zero. If the network switches among a finite number of graphs, which are jointly-connected, then A4) holds.

Adaptive selection of the number of quantization levels

Theorem 1: Suppose A1)-A4) hold. If $h\in(0,1/(2d^*)),\mu<1/(1-h\lambda/(T+1))^{1/2T}$**, and for any** $i\in V,\ j\in N_i$ **the number of quantization levels satisfies**

$$K_{ij}(1)\ge\frac{C_x}{g(0)}-\frac{1}{2},\quad K_{ij}(2)\ge\begin{cases}\zeta_1(h,g(0),g(1)),a_{ij}(1)=1,\\ \zeta_2(h,g(0),g(1)),a_{ij}(1)=0,\end{cases}\quad K_{ij}(t+1)\ge\begin{cases}\kappa_{h,\mu}+\dfrac{\mu(2hd^*+1)}{2}-\dfrac{1}{2},a_{ij}(t)=1,\\ \kappa_{h,\mu}+\eta_{ij}(t),\qquad,a_{ij}(t)=0,\end{cases}\ t=2,3,\ldots$$

where

$$\zeta_1(h,g(0),g(1))=\frac{2hd^*C_\delta+(2hd^*+1)g(0)}{2g(1)}-\frac{1}{2},\ \zeta_2(h,g(0),g(1))=\frac{hd^*C_\delta+(hd^*+K_{ij}(1)+\frac{1}{2})g(0)}{g(1)}-\frac{1}{2},$$

$$\eta_{ij}(t)=\frac{g(t-1)}{g(t)}(hd^*+K_{ij}(t))+\frac{g(t-1)/g(t)-1}{2},\ \kappa_{h,\mu}=\frac{\sqrt{2N}C_\delta hd^*\mu^2(1-\frac{h\lambda}{T+1})^{1/2T}}{g(0)}+\frac{\sqrt{2N}h^2\mu^2(d^*)^2}{1-(1-\frac{h\lambda}{T+1})^{1/2T}\mu},$$

then under the protocol (1), (2) and (3), the closed-loop system satisfies

$$\limsup_{t\to\infty}\frac{\max_{ij}|x_i(t)-x_j(t)|}{g(t)}\le\frac{\sqrt{2N}hd^*\mu^2}{1-(1-\frac{h\lambda}{T+1})^{1/2T}\mu}.$$

Furthermore, if $\lim_{t\to\infty}g(t)=0$**,**

$$\lim_{t\to\infty}x_i(t)=\frac{1}{N}\sum_{j=1}^{N}x_j(0),\ i=1,2,\ldots,N$$

Finite duration of link failures

For any $i=1,2,\ldots,N$, denote the first t such that $a_{ij}(t)=1$ by $t_{ij}(1)$, and

$$t_{ij}(k)=\min\{t:t>t_{ij}(k-1),a_{ij}(t-1)=0,a_{ij}(t)=1\},k=2,3,\ldots,$$
$$s_{ij}(k)=\min\{t:t>t_{ij}(k),a_{ij}(t-1)=1,a_{ij}(t)=0\},k=1,2,\ldots,$$

A5): There is an integer $T_R>0$, such that $t_{ij}(k+1)-s_{ij}(k)\le T_R,i=1,2,\ldots,N,\ j\in N_i$.

Here, $s_{ij}(0)$ is set to zero.

Theorem 2: Suppose A1)-A5) hold. For any integer $K\ge 1$**, denote**

$$\Omega_{K,T_R}=\{(h,\mu)\mid h\in(0,1/(2d^*),\mu\in(1,1/(1-\frac{h\lambda}{T+1})^{1/2T}),\quad\kappa_{h,\mu}+\frac{\mu(2hd^*+1)}{2}\le K+\frac{1}{2},$$
$$\mu^{T_R}K+(\mu hd^*+\frac{\mu-1}{2}+\kappa_{h,\mu})\frac{\mu^{T_R}-1}{\mu-1}\le K+1\}$$

Then the set $\Omega_{h,\mu}$ is nonempty and for any h and $g(t)$ such that $(h,\mu)\in\Omega_{K,T_R}$, and

$$g(0)\ge\frac{C_x}{K+1/2},\ g(1)\ge\frac{2hd^*C_\delta+(2hd^*+1)g(0)}{2K+1},$$

under the protocol (1), (2) and (3) with the number of quantization levels satisfying

$$K_{ij}(1)=K_{ij}(2)=K,$$
$$K_{ij}(t+1)=\begin{cases}K, & a_{ij}(t)=1,\\ K+1, & a_{ij}(t)=0,\end{cases}\ t=2,3,\ldots,$$

The closed-loop system satisfies

$$\lim_{t\to\infty}x_i(t)=\frac{1}{N}\sum_{j=1}^{N}x_j(0),\ i=1,2,\ldots,N$$

Contact: Prof. Lihua Xie, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798. Email: elhxie@ntu.edu.sg