A Solution to the Consensus Problem via Stochastic Mean Field Control

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Large Population Stochastic Consensus Problem

- A Consensus Process is the process of dynamically reaching an agreement between agents of the group .
- We formulate the large population stochastic consensus problem as a dynamic game problem with:
(i) The stochastic dynamics for an individual agent:

$$dz_i(t) = u_i(t)dt + \sigma dw_i(t), \quad t \geq 0, \; 1 \leq i \leq N, \qquad (1)$$

$z_i \in \mathbb{R}$: the state of agent i; $u_i \in \mathbb{R}$: the control input
$\{w_i, 1 \leq i \leq N\}$: independent $(\Omega, \mathbb{B}, P_w)$ Wiener processes.
(ii) The individual Long Run Average (LRA) cost function:

$$J_i^{(N)} \triangleq \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\left[z_i - \frac{1}{N-1}\sum_{k\neq i}^{N} z_k \right]^2 + ru_i^2 \right) dt, \qquad (2)$$

where $1 \leq i \leq N$ and $r > 0$.

The Mean Field Equation System

- In systems with large populations we shall approximate $z_{-i}^* \triangleq \frac{1}{N-1}\sum_{k\neq i}^N z_k$ by a deterministic mass function, z^*.
- Assume that the (empirical distribution of) sequence of initial conditions converges (in the weak sense) to a limiting probability distribution function $F(\cdot)$. Define

$$z^*(t) = \int_{\mathcal{A}} \bar{z}_\alpha(t) dF(\alpha) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N Ez_i, \quad a.s. \; dF,$$

where $\bar{z}_\alpha \triangleq Ez_\alpha$.
- Then the Mean Field equation system is given by:

$$\frac{ds}{dt} = \frac{1}{\sqrt{r}}s + z^*, \qquad (3)$$

$$\frac{d\bar{z}_\alpha}{dt} = -\frac{1}{\sqrt{r}}\bar{z}_\alpha - \frac{1}{r}s, \quad \alpha \in \mathcal{A}, \qquad (4)$$

$$z^*(t) = \int_{\mathcal{A}} \bar{z}_\alpha(t) dF(\alpha), \qquad (5)$$

  - Each individual equation in (4) is indexed by the parameter α associated with initial condition $\bar{z}_\alpha(0)$ which collectively gives the IC for (5). (3) needs no IC.
  - $z^*(\cdot)$ is characterized by the property that it is reproduced as the average of all agents whenever each individual agent optimally tracks $z^*(\cdot)$ by application of the MF control law:

$$u_\alpha = -\frac{1}{\sqrt{r}}z_\alpha - \frac{1}{r}s.$$

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Analysis and Optimality of the Mean Field Control Laws

- Theorem 1: The system (3)-(5) has the unique solution:

$$z^*(t) = z^*(0) \triangleq \int_{\mathcal{A}} z_\alpha(0) dF(\alpha), \quad s(t) = -\sqrt{r}z^*(0), \quad t \geq 0.$$

∎

- The application of the infinite population MF control laws in the finite population case yields

$$u_i^o(t) = \frac{-1}{\sqrt{r}}(z_i(t) - z^*(0)), \quad t \geq 0, \; 1 \leq i \leq N, \qquad (6)$$

and the resulting closed-loop dynamics:

$$dz_i^o(t) = \frac{-1}{\sqrt{r}}(z_i^o(t) - z^*(0))dt + \sigma dw_i(t), \; t > 0, \qquad (7)$$

for $1 \leq i \leq N$, which gives

$$z_i^o(t) = z^*(0) + e^{\frac{-t}{\sqrt{r}}}(z_i(0) - z^*(0)) + \sigma \int_0^t e^{-\frac{(t-\tau)}{\sqrt{r}}} dw_i(\tau). \qquad (8)$$

- Theorem 2: By use of the MF control laws (6) in the dynamic game model (1)-(2), a mean-consensus is asymptotically reached almost surely, *i. e.*

$$\lim_{t\to\infty} \bar{z}_i^o(t) = z^*(0), \qquad 1 \leq i \leq N,$$

with individual asymptotic variance $\sigma^2\sqrt{r}/2$. ∎
- Theorem 3:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N z_i^o(t) = z^*(0), \qquad a.s. \; dF \quad \forall t \geq 0.$$

∎

- Theorem 4: The set of MF control laws (6) generates an almost sure ϵ_N-Nash equilibrium such that for any fixed i, $1 \leq i \leq N$:

$$J_i^{(N)}(u_i^o, u_{-i}^o) - O(\epsilon_i^{(N)}) \leq \inf_{u_i \in \mathcal{U}} J_i^{(N)}(u_i, u_{-i}^o) \leq J_i^{(N)}(u_i^o, u_{-i}^o), \quad a.s.$$

where $\lim_{N\to\infty} \epsilon_i^{(N)} = 0, \; a.s. \; dF$ (if $\sigma = 0$ then $\epsilon_i^{(N)} = 0$). ∎

Localized Feedback in the Deterministic Formulation

- Assume $\sigma = 0$ and agents have no a priori information on $z^*(0)$.
- The observation process of agent i is assumed to be

$$dy_i(t) = \frac{1}{|\mathcal{N}_i(t)|}\sum_{j\in\mathcal{N}_i(t)} z_j(t)dt, \quad t \in [t_{k_i}, t_{k_{i+1}}) \qquad (9)$$

where $\mathcal{N}_i(t) \triangleq \{1 \leq j \leq N | (i,j) \in \mathcal{E}(G_{t_{k_i}})\}, \; t \in [t_{k_i}, t_{k_{i+1}})$.
- Motivated by (7) we set the localized feedback dynamics:

$$dz_i(t) = \frac{-1}{\sqrt{r}}(z_i(t)dt - dy_i(t)), \qquad (10)$$

where $1 \leq i \leq N$, $t \in [t_{k_i}, t_{k_{i+1}})$ and $t_{k_{i+1}} - t_{k_i} = \tau < \infty$.

Localized Feedback in the Deterministic Formulation (Cnt)

- For every $t \geq 0$, (10) may be written in vector form as

$$dz(t) = -\frac{1}{\sqrt{r}}L_t z(t)dt, \quad t \in [t_{k_i}, t_{k_{i+1}}), \qquad (11)$$

where $L_t = I - A_t$ is the time-variant Laplacian of the network topology. (11) gives the standard consensus algorithm.
- Theorem 5 (Based on Ren and Beard (2005)): In case:
(i) the infinite sequence of intervals $[t_k, t_{k+1})$, $k = 1, 2, \cdots$, satisfy the property that the union of graphs $\bigcup_{t_k \leq t < t_{k+1}} G_t$ on each interval $[t_k, t_{k+1})$ has a spanning tree,
(ii) $|\mathcal{N}_i(t)| \to \infty$ for all i and t, we have

$$\lim_{t\to\infty} \lim_{|\mathcal{N}_i(t)|\to\infty} z_i(t) = z^*(0), \qquad 1 \leq i \leq N.$$

∎

Centralized Deterministic Optimal Consensus Problem

- Consider the centralized cost function for a group of N agents in the deterministic case (*i.e.* $\sigma = 0$)

$$J_{cen}^{(N)}(u) \triangleq \sum_{i=1}^N J_i^{(N)}(u)$$

$$\equiv \limsup_{T\to\infty} \frac{1}{T}\int_0^T \{(\tilde{L}^{(N)}z)^T\tilde{L}^{(N)}z + u^T Ru\}dt, \qquad (12)$$

where $z = (z_1, \cdots, z_N)$, $u = (u_1, \cdots, u_N)$, $R = diag(r, \cdots, r)$ and $\tilde{L}^{(N)} = (l_{ij})$ where $l_{ii} = 1$ and $l_{ij} = \frac{1}{N-1}$ for all $1 \leq j \neq i \leq N$.
Observe: $\tilde{L}^{(N)}$ is a Laplacian matrix for a (necessarily) completely connected graph.
- Optimal control law for (12) is given by:

$$\tilde{u}(t) = -R^{-1}\Pi^{(N)}z(t), \qquad (13)$$

where $\Pi^{(N)} = \tilde{L}^{(N)}R^{1/2}$. But (13) gives the standard consensus dynamics for time-invariant completely connected graph:

$$d\tilde{z} = -R^{-1/2}\tilde{L}^{(N)}\tilde{z}dt.$$

- Social Certainty Equivalence: consider the social game:

$$\min_{u_1,\cdots,u_N} \sum_{i=1}^N J_i^{(N)}(u_i), \qquad (14)$$

where $J_i^{(N)}$ is given in (2). Then, in the infinite population case, the SCE methodology gives the same MF system of equations as (3)-(5)!
- For the games (12) and (14), and the optimization problem (2) we have:

Game value = Social value = Centralized value