# Lyapunov-Krasovskii functionals for the study of stability and stabilisation of time-delay systems with application to networked control systems

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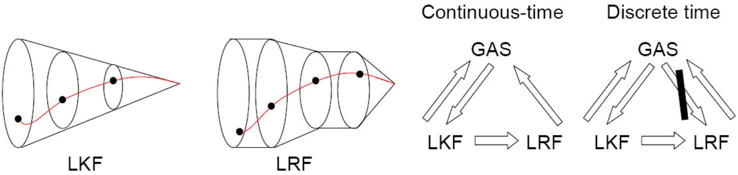
# Contents

# Introduction

## Stability of time-delay systems

- Lyapunov functions for systems without delays.
- Time-delay systems Lyapunov-Razumikhin (LRF) and Lyapunov-Krasovskii functionals (LKF).
- Results for continuous and discrete systems.




## Review work

1. R.H. Gielen, M. Lazar and I.V. Kolmanovsky,"On Lyapunov theory for delay difference inclusions", Proceedings of the American Control Conference 2010.

# The LKF approach

## Goals

- Analyze stability of linear time delay systems [1].
- Robust stability analysis [2].
- Robust controller designs [3].
- Optimal controller designs [4].
- Mixed $H_2/H_\infty$ controllers [4].

## Related publications


1 L. Orihuela, P. Millán, C. Vivas and F.R. Rubio, " Robust stability of nonlinear networked control systems with interval time-varying delay". International Journal of Robust and Nonlinear Control.

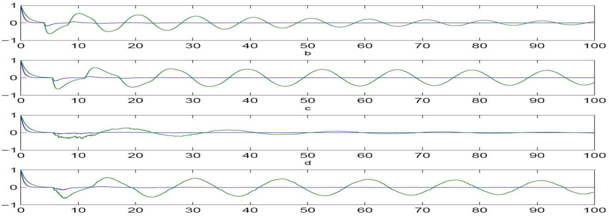
2 P. Millán, L. Orihuela, C. Vivas and F.R. Rubio. "Improved delay-dependent stability for uncertain networked control systems with induced time-varying delays". 1st IFAC Workshop on Estimation and Control of Networked Systems, 2009.

3 J. Arriaga, P. Millán, I. Jurado, C. Vivas, F.R. Rubio, "Application of Network-based Robust Control to a Personal Pendulum Vehicle", European Control Conference ECC 2009

4 P. Millán, Luis Orihuela, C. Vivas and F.R. Rubio, "An optimal control $L_2$-gain disturbance rejection design for networked control systems". ACC 2009.

## Non reliable communication channel

- Sampling.
- Randomly time-varying delays.
- Packages dropouts.
- Other: quantization, energy aware, etc.




## Posible consequences

- Degradation of the control performance
- "Expensive" control
- Unstable behaviours

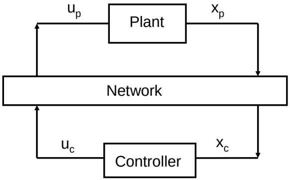
# Modelling Networked Control Systems (NCSs) as Time Delay Systems (TDSs)

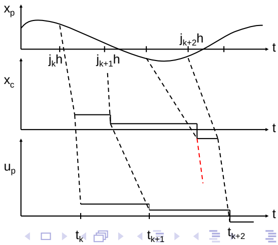
## Scope

We work with Linear Time Invariant Systems (LTI) with differentiated uncertainties, disturbances and non-ideal networked links in sensor-to-controller and controller-to-actuator paths

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + B_\omega \omega(t), \\
z(t) &= Cx(t) + Du(t), \\
x(t_0) &= x_0,
\end{aligned}
$$

- Sensor nodes sample data in a time-driven manner at time instants $t = j_k h$ such that $\{j_1, j_2, j_3, \dots\} \subseteq \{1, 2, 3, \dots\}$ and $j_k < j_{k+1}$.
- $t \in [t_k, t_{k+1})$ time intervals with constant control input.



# SYSTEM UNCERTAINTIES

## Parametric uncertainties

$$[A, B] \to [A^* + \Delta A(t), B^* + \Delta B(t)]$$

$$\begin{aligned} \Delta A(t) &= G_1 F_1(t) E_1, \\ \Delta B(t) &= G_2 F_2(t) E_2 \end{aligned}$$

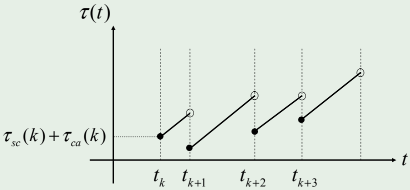
## Polytopic uncertainties

$\Omega = [A \quad B] / \Omega \in conv\{\Omega_j, j = 1, \ldots, N\}$,
$\Omega_j = [A^{(j)} \quad B^{(j)}]$. $\Omega = \sum_{i=1}^{12} \sum_{j=1}^{N} f_j \Omega_j$,
$0 \leq f_j \leq 1$, $\sum_{j=1}^{N} f_j = 1$.

## Close loop NCS model ($u = Kx$ )

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t - \tau(t)) + B_\omega \omega(t), \quad \forall t \in [t_k, t_{k+1}), \\ z(t) &= Cx(t) + DKx(t - \tau(t)), \quad \forall t \in [t_k, t_{k+1}), \\ x(t) &= \phi(t), \quad t \in [t_0 - \tau_M, t_0], \end{aligned}$$

where $\tau(t) = t - t_k + \tau_{sc}(k) + \tau_{ca}(k)$




Sampling period $h$, Round-Trip delay $\tau_{RT}$ and consecutive package losses $N_p$ bounded imply that $\tau(t)$ is bounded: $\underline{\tau}_{RT} \leq \tau(t) \leq \overline{\tau}_{RT} + (h+1)\overline{N}_p$

# General procedure

## Notation

- $\zeta$ Set of network parameters: $\tau_M, \tau_m, N_p, h$
- $\xi^T(t) = [x^T(t), x^T(t-\tau(t)), ...]$
- $V(t,\zeta)$ Candidate to be a Lyapunov-Krasovskii functional
- $J = \int_{t_0}^{\infty} [\xi^T(t)\Phi(K)\xi(t)]dt$ Cost function.

**LINEAR MATRIX INEQUALITIES FOR DIFFERENT CONTROL PROBLEMS**

## System stability

$$\frac{d}{dt}V(t,\zeta) = \epsilon^T(t)\Xi(K,\zeta)\epsilon(t), \quad \Xi(K,\zeta) < 0$$

## Constraint on the system $L_2$ gain

$$\frac{d}{dt}V(t,\zeta) = \epsilon^T(t)\Xi\epsilon(t) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t), \quad \Xi(K,\zeta) < 0$$

## Optimal control

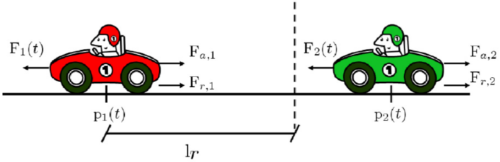
$$\begin{array}{rrcl} & \min_{K} & \alpha, & \\ \text{subject to} & \alpha\Xi(K,\zeta) & < & -\Phi(K) \\ & \alpha & > & 0, \quad \alpha \in \mathbb{R} \end{array}$$

## Technical arguments

1. Schur complement
2. S-procedure
3. Finsler´s lemma
4. Leibnitz-Newton equation
5. Jensen´s inequality
6. Moon´s inequality
7. Slack matrices
8. Polytopic descriptions
9. Cone complementary algorithm

Control problems can be written in terms of **Linear Matrix Inequalities**

## Application of the LK Approach to design $H_2/H_\infty$ controllers




### Vehicle tracking system:

$$\begin{aligned} e(t) &= p_1(t) - p_2(t) - l_r, \\ \dot{e}(t) &= v_1(t) - v_2(t) = y(t), \\ \dot{y}(t) &= a_1(t) - a_2(t). \end{aligned}$$

Forces equilibrium:
$F_i(t) - F_{a,i}(t) - F_{r,i}(t) = m_i a_i(t), \quad i = 1, 2.$
Aerodynamical drag: $F_a(t) = \frac{1}{2} c_a A_T \rho_{aire} v^2(t)$,
Rolling friction: $F_r(t) = c_r mg \cos(\alpha(t))$.

### Parameters

- $c_a$, $c_r$: aerodynamic and tire-road drag coefficients, respectively.
- $A_T$: vehicle's aerodynamic cross-section.
- $\rho_{air}$: air density.
- $m$: vehicle mass.
- $g$: gravity constant.
- $\alpha(t)$: road slope angle.

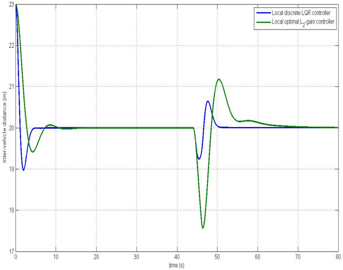
### System's linearization

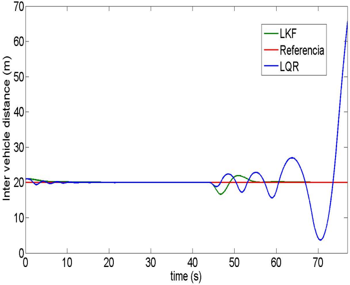
$$\frac{d}{dt}\begin{pmatrix} \int e(t) \\ e(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -c_2 c_1 \end{pmatrix} \begin{pmatrix} \int e(t) \\ e(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -c_3 \end{pmatrix} F_2(t) + \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} F_1(t)$$

Control Law:

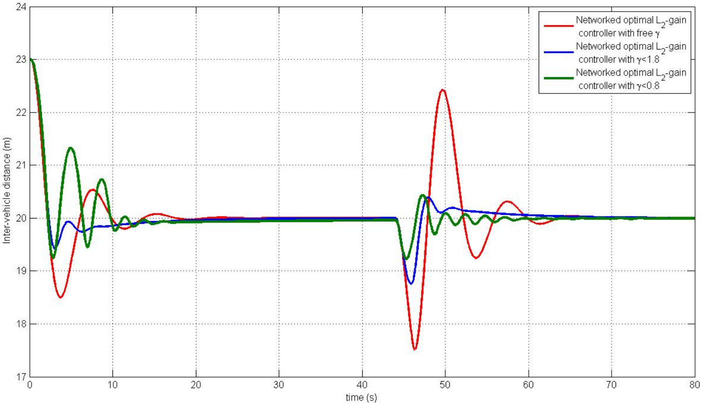
$$F_2(t) = K \begin{pmatrix} \int e(t-\tau(t)) \\ e(t-\tau(t)) \\ y(t-\tau(t)) \end{pmatrix}$$

# Local and Networked Control

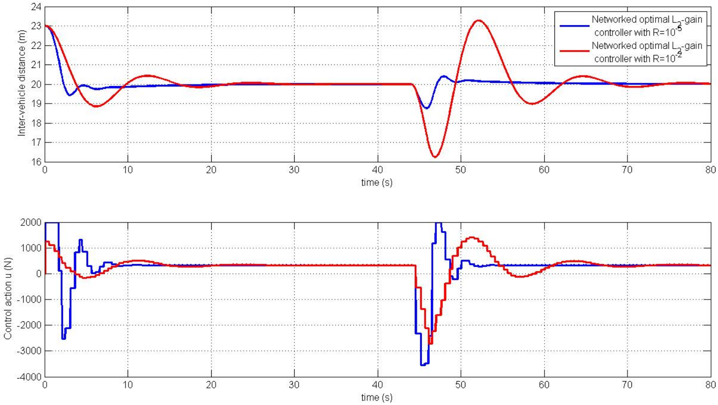


## Response with network effects for various disturbance attenuation levels



# Trading off performance vs. control effort



# Conclusions

## Conclusions

- Closing control loops over communication networks requires the participation of new analysis and design techniques.
- The LK approach makes possible the design of controller considering different specification.
- The problems are solved using LMI.
- Simulation examples shows the good performance of the controllers designed with this approach.

Thank you for your attention.