# Performance Comparison of the Distributed Extended Kalman Filter and Markov Chain Distributed Particle Filter


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## Introduction

We compare consensus-based distributed filters for multiple range-only sensors tracking vehicles moving with nonlinear dynamics. The filters are:

1. MCDPF: Markov Chain Distributed Particle Filter [LW09].
2. DEKF: Distributed Extended Kalman Filter [OSS05]: EKF written in information filter form with consensus on the information matrices and vectors.
3. RDEKF: Regularized Distributed Extended Kalman Filter: same as DEKF but with the condition number of the information matrix constrained within $[1, 10^6]$.

### System Model

Assume a nonlinear system with additive Gaussian noise:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t) + q_t$$
$$\mathbf{z}_t = h(\mathbf{x}_t) + r_t$$

where $\mathbf{x}_t \in \mathbf{R}^n$, $\mathbf{z}_t \in \mathbf{R}^p$, and $w_t \sim \mathcal{N}(0, Q_t)$, $r_t \sim \mathcal{N}(0, R_t)$ are process and measurement noises respectively. State estimates are

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{E}(\mathbf{x}_t|\mathbf{z}_{1:t-1}), \quad \hat{\mathbf{x}}_{t|t} = \mathbf{E}(\mathbf{x}_t|\mathbf{z}_{1:t})$$

## Particle Filters

### Centralized Particle Filter (CPF)

The posterior distribution at time $t-1$, $\pi_{t-1|t-1}(d\mathbf{x}_{t-1})$, approximated by $N$ particles $\{\mathbf{x}^i_{t-1}\}_{i=1}^N$ is

$$p(\mathbf{x}_{t-1}|\mathbf{z}_{1:t-1}) \triangleq \pi_{t-1|t-1}(d\mathbf{x}_{t-1})$$
$$\approx \pi^N_{t-1|t-1}(d\mathbf{x}_{t-1}) = \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{x}^i_{t-1}}(d\mathbf{x}_{t-1})$$

**Prediction step:** $\tilde{\mathbf{x}}^i_t \sim \frac{1}{N}\sum_{i=1}^N K(\mathbf{x}_t|\mathbf{x}^i_{t-1})$ gives

$$p(\mathbf{x}_t|\mathbf{z}_{1:t-1}) \triangleq \pi_{t|t-1}(d\mathbf{x}_t) \approx \tilde{\pi}^N_{t|t-1}(d\mathbf{x}_t) = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{\mathbf{x}}^i_t}(d\tilde{\mathbf{x}}_t)$$

**Measurement update step:** $\rho(\mathbf{z}_t|\mathbf{x}_t)$ is the transition probability density of a measurement $\mathbf{z}_t$ given the state $\mathbf{x}_t$.

$$\tilde{\pi}^N_{t|t}(d\mathbf{x}_t) \triangleq \frac{\rho(\mathbf{z}_t|\mathbf{x}_t)\tilde{\pi}^N_{t|t-1}(d\mathbf{x}_t)}{\int_{\mathbb{R}^n}\rho(\mathbf{z}_t|\mathbf{x}_t)\tilde{\pi}^N_{t|t-1}(d\mathbf{x}_t)\,d\mathbf{x}_t}$$
$$= \frac{\sum_{i=1}^N \rho(\mathbf{z}_t|\tilde{\mathbf{x}}^i_t)\delta_{\tilde{\mathbf{x}}^i_t}(d\tilde{\mathbf{x}}_t)}{\sum_{i=1}^N \rho(\mathbf{z}_t|\tilde{\mathbf{x}}^i_t)} = \sum_{i=1}^N w^i_t \delta_{\tilde{\mathbf{x}}^i_t}(d\tilde{\mathbf{x}}_t)$$

**Resampling step:** to avoid degeneracy problems, giving

$$\pi^N_{t|t}(d\mathbf{x}_t) = \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{x}^i_t}(d\mathbf{x}_t)$$

### Distributed Approach (MCDPF)

Each sensor in a network has a set of particles to approximate the posterior distribution. Particles hop along network edges ($k_{\rm mc}$ hops per measurement update). Assuming independent measurement noises, measurement updates can be applied incrementally as each particle passes through a sensor. Prediction and resampling are done per-sensor.

**Theorem 1.** *Consider a connected sensor network with measurements at different nodes conditionally independent given the true state. Then the estimated distribution of the MCDPF algorithm converges weakly to the estimated distribution of the CPF as the number of Markov chain steps $k_{\rm mc}$ per measurement goes to infinity. That is,*

$$\lim_{k_{\rm mc}\to\infty} \pi^N_{t|t,k} = \pi^N_{t|t}.$$

*pointwise.*

*Proof.* See [LW09, Section C]. □

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**Algorithm 1** Markov Chain Distributed Particle Filter

**Initialization: for** $j = 1$ to $m$ **do**
  $\{\mathbf{x}^i_{j,0}\}_{i=1}^N \sim p(\mathbf{x}_0)$, $\{w^i_{j,0}\}_{i=1}^N = \frac{1}{N}$
**Prediction: for** $j = 1$ to $m$ **do**
  $\{\tilde{\mathbf{x}}^i_{j,t}\}_{i=1}^{N(j)} \sim p\left(\mathbf{x}_t|\{\mathbf{x}^i_{j,t-1}\}_{i=1}^{N(j)}\right)$, $\{\tilde{w}^i_{j,t}\}_{i=1}^{N(j)} = 1$
**Measurement update (Markov chain random walk):**
  **for** $k_{\rm mc}$ iterations **do**
   move $\{\tilde{\mathbf{x}}^i_{\cdot,t}\}_{i=1}^{N(j)}$, $\{\tilde{w}^i_{\cdot,t}\}_{i=1}^{N(j)}$ randomly along edges
   **for** $j = 1$ to $m$ **do**
    $\{\tilde{x}^i_{j,t}\}_{i=1}^{N(j)} = \bigcup_{l\in N_j}\{\tilde{x}^i_{l,t}\}_{i\in\mathcal{I}_{l\to j}}$
    $\{\tilde{w}^i_{j,t}\}_{i=1}^{N(j)} = \bigcup_{l\in N_j}\{\tilde{w}^i_{l,t}\}_{i\in\mathcal{I}_{l\to j}}$
    $\{\tilde{w}^i_{j,t}\}_{i=1}^{N(j)} \leftarrow \{\tilde{w}^i_{j,t}\}_{i=1}^{N(j)} \times \rho_j\left(\mathbf{z}_{j,t}|\{\tilde{\mathbf{x}}^i_{j,t}\}_{i=1}^{N(j)}\right)^{\frac{2|E(G)|}{kd(j)}}$
   **end for**
  **end for**
**Resample: for** $j = 1$ to $m$ **do**
  resample $\{\mathbf{x}^i_{j,t}\}_{i=1}^{N(j)}$ according to $\{\tilde{w}^i_{j,t}\}_{i=1}^{N(j)}$
  set weights $\{w^i_{j,t}\}_{i=1}^{N(j)} = \frac{1}{N(j)}$

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## Performance Comparison

Four fixed sensor stations (black triangles) perform range-only measurements to vehicles and can each communicate with two other sensors (black dotted lines). The vehicles move according to a nonlinear flocking control law [TJP03] that includes global cooperation and local collision avoidance.

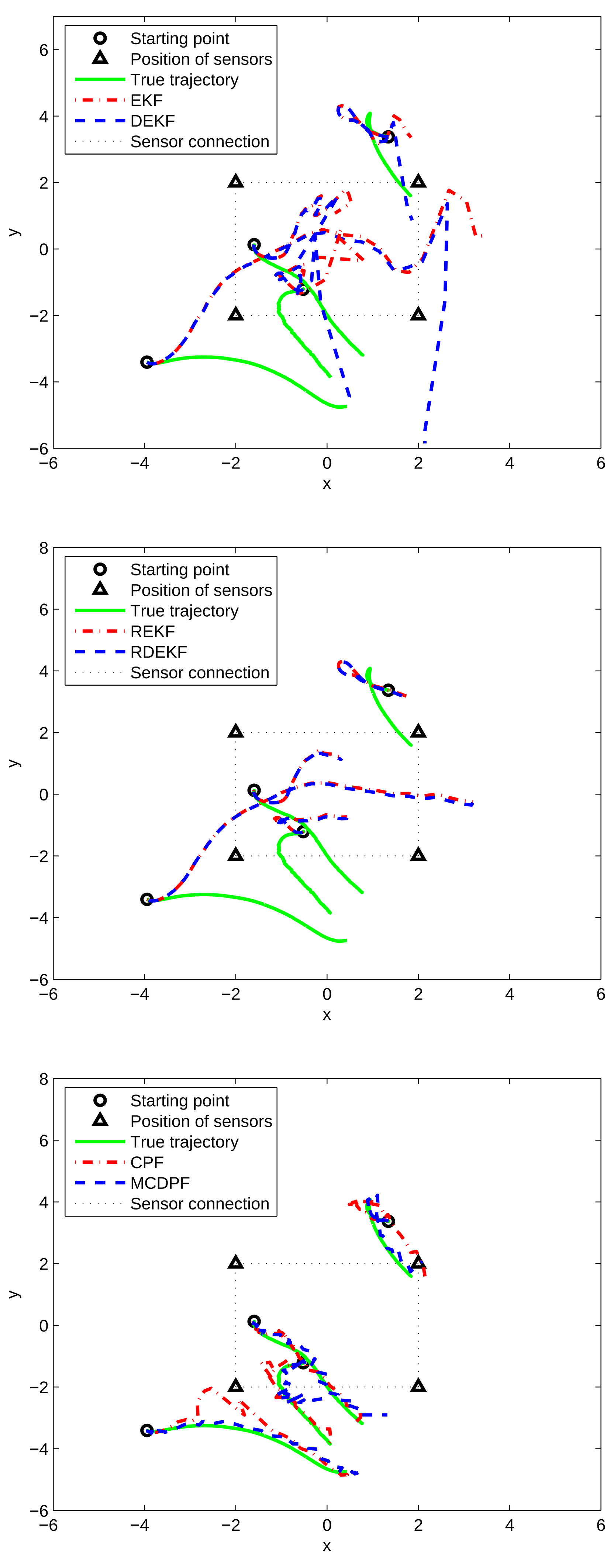


**Figure 1:** Estimated states of 4 vehicles with $\Delta t = 0.1$.

Estimator performance is evaluated by comparing the root mean squared error (RMSE) of the estimated mean. An estimator is said to *diverge* if the RMSE is higher than the static estimator that always returns the initial condition.

### Changing sample rate and system size

| **4 vehicles** | $\Delta t = 0.05$ | | $\Delta t = 0.1$ | |
|---|---|---|---|---|
| Algorithm | RMSE | Diverge | RMSE | Diverge |
| EKF | 0.124 | 0% | N/A | 49.7% |
| DEKF | 0.173 | 0.1% | N/A | 62.3% |
| Regularized EKF | 0.116 | 0% | 1.20 | 2.1% |
| Regularized DEKF | 0.158 | 0% | 2.54 | 2.3% |
| CPF | 0.192 | 0% | 0.287 | 0% |
| MCDPF | 0.745 | 0% | 0.801 | 0% |

| **10 vehicles** | $\Delta t = 0.05$ | | $\Delta t = 0.1$ | |
|---|---|---|---|---|
| Algorithm | RMSE | Diverge | RMSE | Diverge |
| EKF | N/A | 100% | N/A | 100% |
| DEKF | N/A | 100% | N/A | 100% |
| Regularized EKF | 1.57 | 2.9% | 1.71 | 8% |
| Regularized DEKF | 5.83 | 6.1% | 12.7 | 16.7% |
| CPF | 0.243 | 0% | 0.499 | 0% |
| MCDPF | 0.481 | 0% | 0.742 | 0% |

The filter parameters were always $N = 100$ particles for CPF and MCPDF, $k_{\rm mc} = 10$ Markov chain iteration steps for MCDPF, and $k_{\rm con} = 2$ consensus iteration steps for (R)DEKF.

### Convergence of distributed filters

Information exchange bandwidth (amount of data to be sent per unit time) at each node:

$$\text{BW}_{\rm CPF} = \frac{1}{\Delta t}N(n+1)$$
$$\text{BW}_{\rm MCDPF} = \frac{1}{\Delta t}k_{\rm mc}\left(\frac{N}{m}(n+1)\right)$$

for $n$ system dimensions and $m$ sensors.

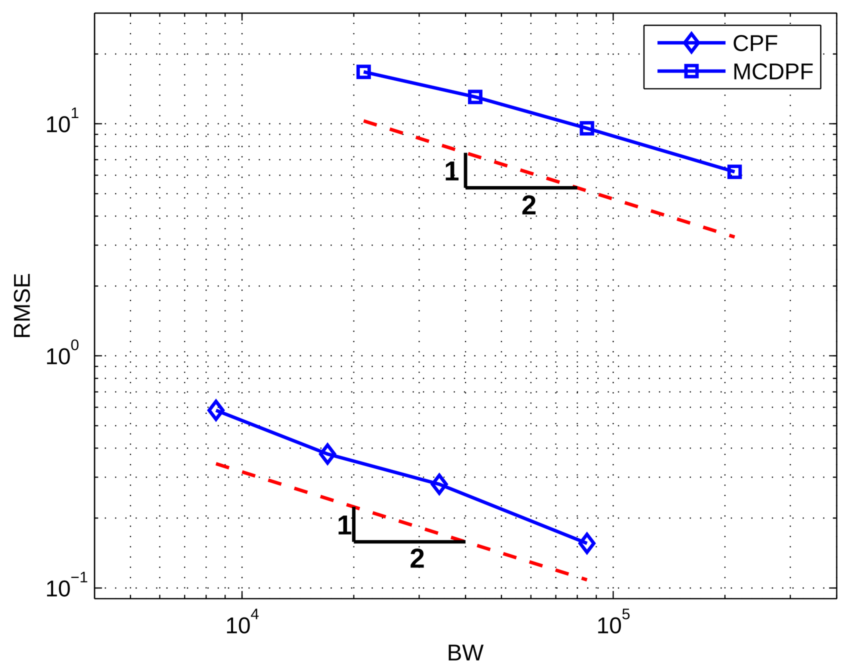


**Figure 2:** Convergence as $N \to \infty$ (CPF) and $k_{\rm mc} \to \infty$ (MCDPF).

We observe the proven limiting rates of $1/\sqrt{\text{BW}}$ for the particle filters.

## Conclusions

For sufficiently small measurement timesteps and sufficiently small systems dynamics, the Kalman-type filters outperform the particle filters (in particular, RDEKF beats MCDPF). However, in all other cases MCDPF beats RDEKF. We summarize this as:

| | $\Delta t = 0.05$ | $\Delta t = 0.1$ |
|---|---|---|
| 4 vehicles | RDEKF better | MCDPF better |
| 10 vehicles | MCDPF better | MCDPF better |

Additionally, we found that regularization is necessary to obtain reasonable behavior from Kalman-type filters for this system.

### Bibliography


[LW09] S. H. Lee and M. West. Markov Chain Distributed Particle Filters (MCDPF). In *48th IEEE CDC/28th CCC*, pages 5496–5501, 2009.

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