Controlling Four Agent Formations

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1 The Problem Context

Making a *four agent* formation take up a shape specified by inter-agent distances

Background Aspects

- Single Integrator Model with point agent
- Formation orientation is irrelevant
- Relative positions of neighbors of an agent are constructible (in agent's own coordinate basis) from the six interagent distances
- Decentralized control law proposed by e.g. Krick, Broucke and Francis [1] is used, using error between *square* of actual distances and desired distances and relative positions of neighbors.
- Law is a gradient descent law.

Why Study Just 4 agents?

There are unsolved problems with n > 3 agents. They may be easier to solve in the first instance for 4 agents. They have been solved for 3 agents. [2,3]

Why require control just of shape and not also orientation?

In many applications, either

- orientation will not be important (e.g. circling a target),

or

- orientation will be changed while formation shape is to be preserved (e.g. moving a robot formation from A to B while carrying a big load).

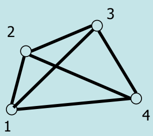
2 The Key Technical Issue

Krick et al found that incorrect equilibria exist.

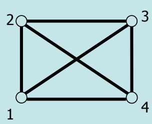
Does this mean their algorithm is unsafe or invalidated?

Our New Contributions

- Demonstration that the incorrect equilibiria identified by Krick, Broucke and Francis are all **saddle points**.
- You can mistake them for a stable equilibrium with a carefully chosen initial condition, and a very low noise that does not throw you off the equilibrium
- Easy means of checking stability of an equilibrium
- Demonstration that for all reasonable formation shapes, incorrect equilibria are likely to occur
- Demonstration that for rectangular formations, there are two incorrect equilibria
  - ➢They are rectangular
  - ➢They are saddles



Actual lengths $d_{ij}(t)$



Desired lengths $\bar{d}_{ij}$

3 Equations of Motion of K_4 graph

$$
\begin{aligned}
p &= [p_1^T, p_2^T, p_3^T, p_4^T]^T \in \mathcal{R}^8 \ \text{(position coordinates)} \\
e_{ij}(p) &= d_{ij}^2 - \bar{d}_{ij}^2(p) \ \text{(errors in squared distances)} \\
V(p) &= \frac{1}{2}||e(p)||^2 \ \text{(Lyapunov function)} \\
\dot{p} &= -\nabla V(p)^T \ \text{(steepest descent law for Lyapunov function)} \\
&= -[J_e(p)]^T e(p) \ \ (J_e(p) \text{ is rigidity matrix})
\end{aligned}
$$

Above is standard. Another description is useful:

$$
\begin{aligned}
E(p) &:= \begin{bmatrix} e_{12}+e_{13}+e_{14} & -e_{12} & -e_{13} & -e_{14} \\ -e_{12} & e_{12}+e_{23}+e_{24} & -e_{23} & -e_{24} \\ -e_{13} & -e_{23} & e_{13}+e_{23}+e_{34} & -e_{34} \\ -e_{14} & -e_{24} & -e_{34} & e_{14}+e_{24}+e_{34} \end{bmatrix} \\
\dot{p} &= -(E(p) \otimes I_2)p
\end{aligned}
$$

4 Defining Equilibria

p^* is an equilibrium if and only if

$$\nabla V(p^*) = (E(p^*) \otimes I_2)p^* = J_e^T(p^*)e(p^*) = 0$$

The Hessian of $V(p)$ is nonnegative at stable but not saddle equilibria:

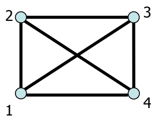
$$\nabla^2 V(p^*) = H_V(p^*) = 2J_e^T(p^*)J_e(p^*) + E(p^*) \otimes I_2$$

Note that p^* is a correct equilibrium iff $e_{ij} = 0 \ \forall i, j$.
Then $E(p^*) = 0$ and $\nabla^2 V(p^*) = 2J_e^T(p^*)J_e^T(p^*) \geq 0$.

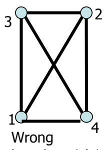
5 What did Krick et al [1] find?

They found that **for a certain rectangular formation, there was an equilibrium which was also rectangular, but not a correct equilibrium.**

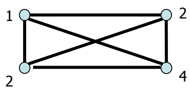
Using above definitions **we can show**:



Desired lengths $\bar{a}, \bar{b}$



Wrong lengths a^*, b^*



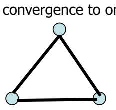
Wrong lengths a^*, b^*

Incorrect equilibria are provably saddles with

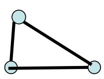
$$
\begin{aligned}
&[\text{Tall}] \ a^{*2} = \bar{a}^2 + \bar{b}^2/3, \ b^{*2} = \bar{b}^2/3 \\
&[\text{Wide}] \ a^{*2} = \bar{a}^2/3, \ b^{*2} = \bar{a}^2/3 + \bar{b}^2
\end{aligned}
$$

6 What happens with 3 agents?

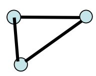
With an initial formation that is a triangle of nonzero area, convergence to one of two congruent possibilities occurs.



Actual lengths $d_{ij}(t)$



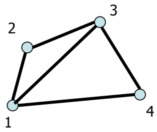
Desired lengths $\bar{d}_{ij}$



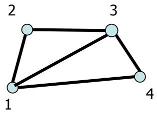
Desired lengths $\bar{d}_{ij}$

If the initial formation is collinear, convergence occurs to a collinear incorrect equilibrium which is a saddle point.

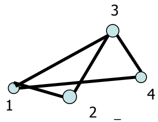
7 What happens with 4 agents and five, not six, nominated distances?



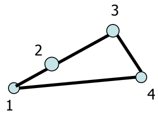
Actual lengths $d_{ij}(t)$



Desired lengths $\bar{d}_{ij}$



Desired lengths $\bar{d}_{ij}$



Incorrect lengths d^*_{ij}

Disregarding complete reflections, there are two noncongruent correct equilibria and at least one saddle point incorrect equilibrium. Use of six distances instead of five eliminates possibility of noncongruent correct equilibria—but leaves the possibility of incorrect equilibria.

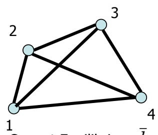
8 Incorrect equilibria in general K_4 graph

An algebraic condition

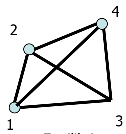
At an incorrect equilibrum defined by coordinates p*, the matrix E(p*) has rank one with a single negative eigenvalue.

A nonobvious consequence

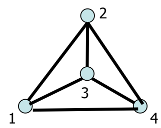
All incorrect equilibria are **twisted** relative to the associated correct equilibrium:



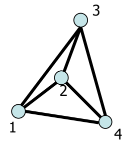
Correct Equilibrium $\bar{d}_{ij}$



Incorrect Equilibrium d^*_{ij}



Correct Equilibrium $\bar{d}_{ij}$



Incorrect Equilibrium d^*_{ij}

Inequalities

Depending on the vertex numbering, certain inequalities always hold. E.g. for the triangles above,

$$
\begin{aligned}
d_{12}^* > \bar{d}_{12}, d_{23}^* > \bar{d}_{23}, d_{24}^* > \bar{d}_{24} \\
d_{13}^* < \bar{d}_{13}, d_{14}^* < \bar{d}_{14}, d_{34}^* < \bar{d}_{34}
\end{aligned}
$$

Understanding this result uses **tensegrity theory [4]**

An odd observation

If an incorrect equilibrium is a parallelogram rather than a rectangle, the correct formation is necessarily another parallelogram or an isosceles trapezoid.

9 Some open questions and conjectures

- **Are all incorrect equilibria always saddle points?**
- What is the number of incorrect equilibria to be expected for a generic formation?
- Is there a simple way to adjust the control law to exclude the possibility of any incorrect equilibria occurring?
  - •Might this require introduction of a third spatial dimension?
  - •Could techniques based on Morse Theory establish that for a two dimensional problem, there are always incorrect equilibria no matter what form of control law is used, or no matter what form of control law is used when it is based on a gradient descent of a smooth potential function?
- When the equilibrium shape is a rectangle, are incorrect equilibria necessarily also rectangles?
- What are the generalizations to more complex formations?

References

[1] Krick, I., Broucke, M. and Francis, B., **Stabilization of infinitesimally rigid formations of multi-robot networks,** International Journal of Control, vol. 82, 2009, pp. 423-439.

[2] Anderson, B.D.O., Yu, C., Dasgupta, S., Morse, A.S., **Control of a three- coleader formation in the plane,** Systems and Control Letters, vol. 56, 2007, pp. 573-578

[3] Cao, M., Morse, A.S., Yu, C., Anderson, B.D.O., and Dasgupta, S. **Maintaining a directed, triangular formation of autonomous agents,** Communications and Information Systems, vol. 11, 2011 to appear.

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