Stochastic Average Consensus Filter for Distributed HMM Filtering: Almost Sure Convergence

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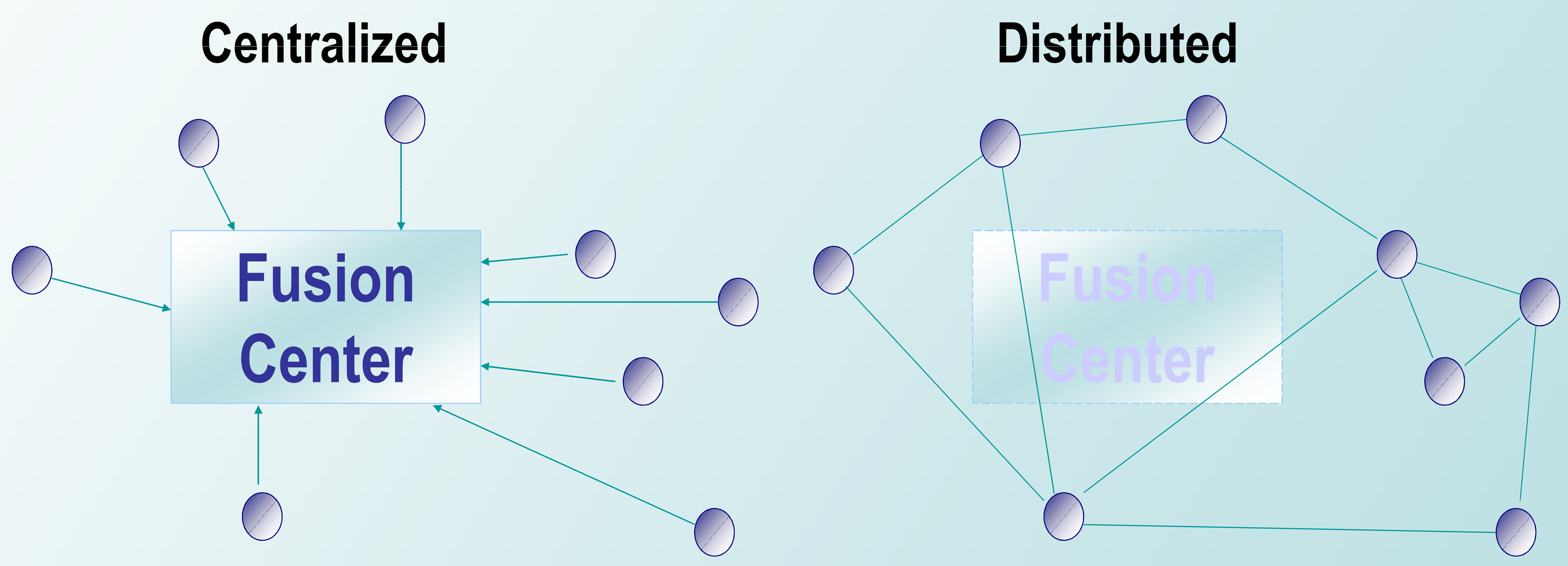
Introduction

❑ **Objective:** To design a *distributed* state estimation algorithm for hidden Markov models using average consensus schemes.

❑ **Motivation:** Scalability with the size of the network, energy efficiency in terms of message exchange, robustness, and efficiency in computation.



❑ **Contribution:** Convergence analysis of a dynamic average consensus algorithm used in a distributed HMM filter

Distributed Filtering Model

- Markov process $\{X_k\}_{k=0}^{\infty}$ with state space S observed by n sensors with observation densities $P(y_k^j \in dy | X_k = \ell) = f_\ell^j(y_k^j)\nu(dy)$
- Centralized filter $\bar{\mathbf{v}}_k = [\bar{v}_k(\ell)]_{\ell \in S}$, $\bar{v}_k(\ell) = P(X_k = \ell | Y_k^1 \cdots Y_k^n)$, where

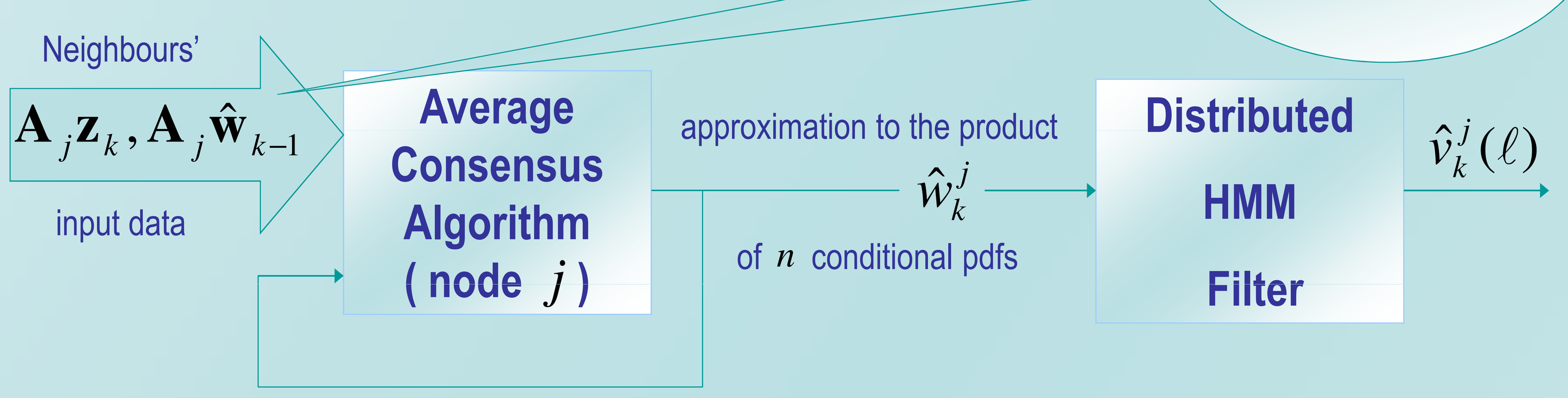
$$\bar{v}_k(\ell) = \prod_{m=1}^{n} f_\ell^m(y_k^m) \sum_{i=1}^{s} x_{i\ell}\bar{v}_{k-1}(i)$$ and by taking log we have :

$$\Rightarrow \log \bar{v}_k(\ell) = -n\bar{w}_k(\ell) + \log \sum_{i=1}^{s} x_{i\ell}\bar{v}_{k-1}(i) \qquad \bar{w}_k(\ell) = \frac{1}{n}\sum_{m=1}^{n} -\log f_\ell^m(y_k^m)$$

- Each node j can compute an approximation $\hat{w}_k^j(\ell)$ to the average quantity $\bar{w}_k(\ell)$ using a *dynamic* average consensus algorithm by exchanging appropriate messages *only with its neighboring nodes*.
- Distributed filter at node j for the state value $\ell \in S$

$$\hat{v}_k^j(\ell) = \exp(-n\hat{w}_k^j(\ell)) \sum_{i=1}^{s} x_{i\ell}\hat{v}_{k-1}^j(i)$$

$$\hat{\mathbf{w}}_k = [\hat{w}_k^j]_{j \in N} \qquad \mathbf{z}_k^\ell = [-\log f_\ell^j(y_k^j)]_{j \in N}$$



- It is clear that only for a complete graph $\hat{w}_k^j(\ell) = \bar{w}_k(\ell)$. For other topologies, without the knowledge of all the sensors' measurements and distribution models at every node, each node may only be able to find an approximation to the centralized filter. This paper is a step towards answering this question:
- ***Question*** : *How close is the distributed filter to the centralized one?*

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Stochastic Approximation Consensus Algorithm

- We consider the following average consensus algorithm proposed by Olfati-Saber and Shamma (2005) : $\hat{\mathbf{w}}_k = \hat{\mathbf{w}}_{k-1} + \rho(\mathbf{\Lambda}\hat{\mathbf{w}}_{k-1} + \mathbf{\Gamma}\mathbf{z}_k)$

where the matrices are defined $\mathbf{\Lambda} = -(\mathbf{I} + \mathbf{D} + \mathbf{L}) \quad \mathbf{\Gamma} = \mathbf{I} + \mathbf{A}$

$\mathbf{A}$ is the adjacency matrix, and $\mathbf{L}$ is the Laplacian matrix of the graph.

- We define the error $\boldsymbol{\eta}_k = \hat{\mathbf{w}}_k - \bar{\mathbf{w}}_k$ with the following dynamics

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + \rho Q(\boldsymbol{\eta}_k, \mathbf{z}_{k+1}, \mathbf{z}_k)$$

- In this paper, we show that asymptotically $\boldsymbol{\eta}_k$ converges $P-$a.s. to a small neighbourhood of the origin.
- We use ODE technique in stochastic approximation.
- First, we need to show stochastic stability of $\boldsymbol{\eta}_k$ using perturbed stochastic Lyapunov function.
- Define the mean ODE $\dot{\boldsymbol{\eta}}_\bullet(t) = \bar{\mathbf{Q}}(\boldsymbol{\eta}_\bullet) \qquad \boldsymbol{\eta}_\bullet(0) = \boldsymbol{\eta}_0$, where

$$\bar{\mathbf{Q}}(\boldsymbol{\eta}_\bullet) = \lim_{k \to \infty} EQ(\boldsymbol{\eta}_\bullet, \mathbf{z}_{k+1}, \mathbf{z}_k)$$

- For an irreducible and aperiodic Markov chain with absolutely positive densities, there exists s finite $\bar{\mathbf{Q}}(\boldsymbol{\eta}_\bullet)$.

Sketch of the proof

There exists a perturbed stochastic Lyapunov function which has supermartingale property

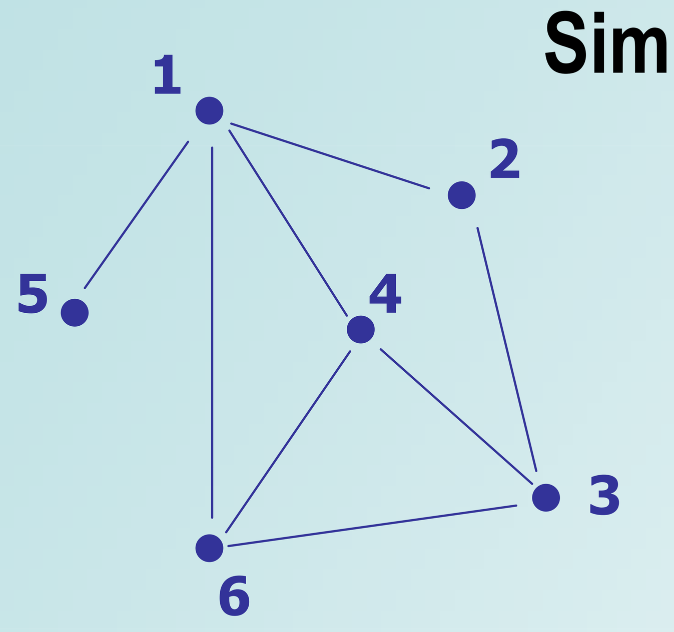
↓

The error $\boldsymbol{\eta}_k$ visits some compact set infinitely often $P-$w.p.1

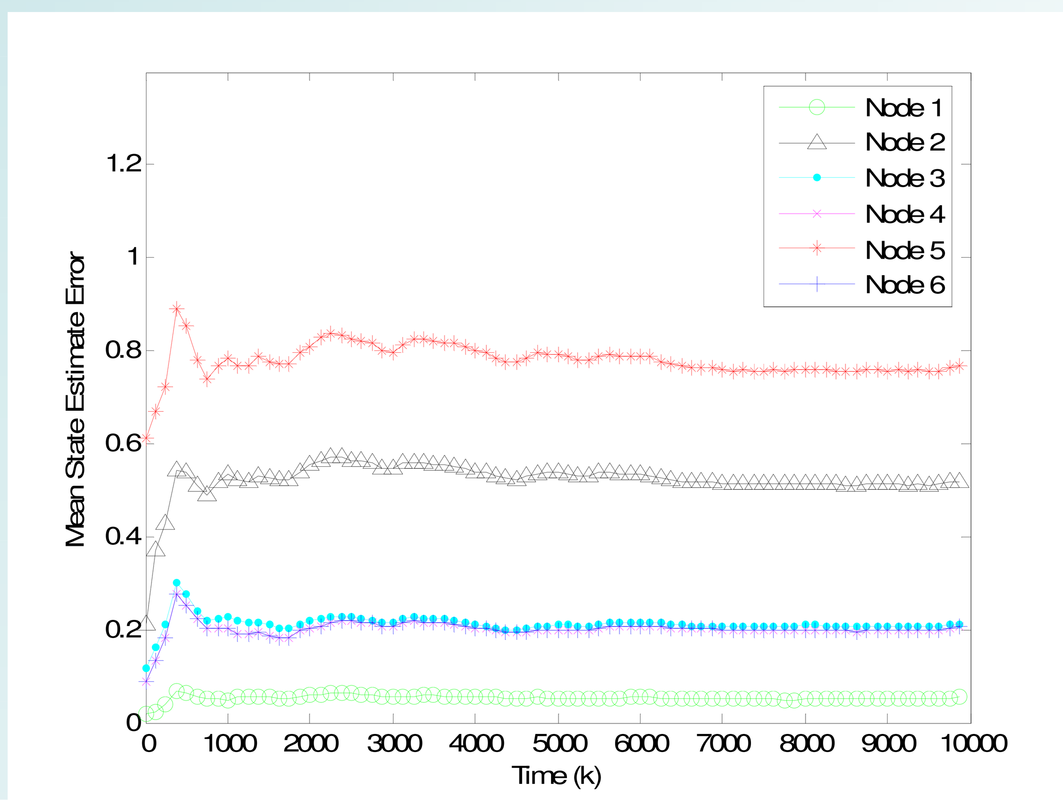
↓

Using the ODE method, it is shown that asymptotically $\boldsymbol{\eta}_k$ starting at the recurrence times when $\boldsymbol{\eta}_k$ enters the compact set converges to the largest bounded invariant set contained in the compact set. The mean ODE needs to be globally asymptotically stable.

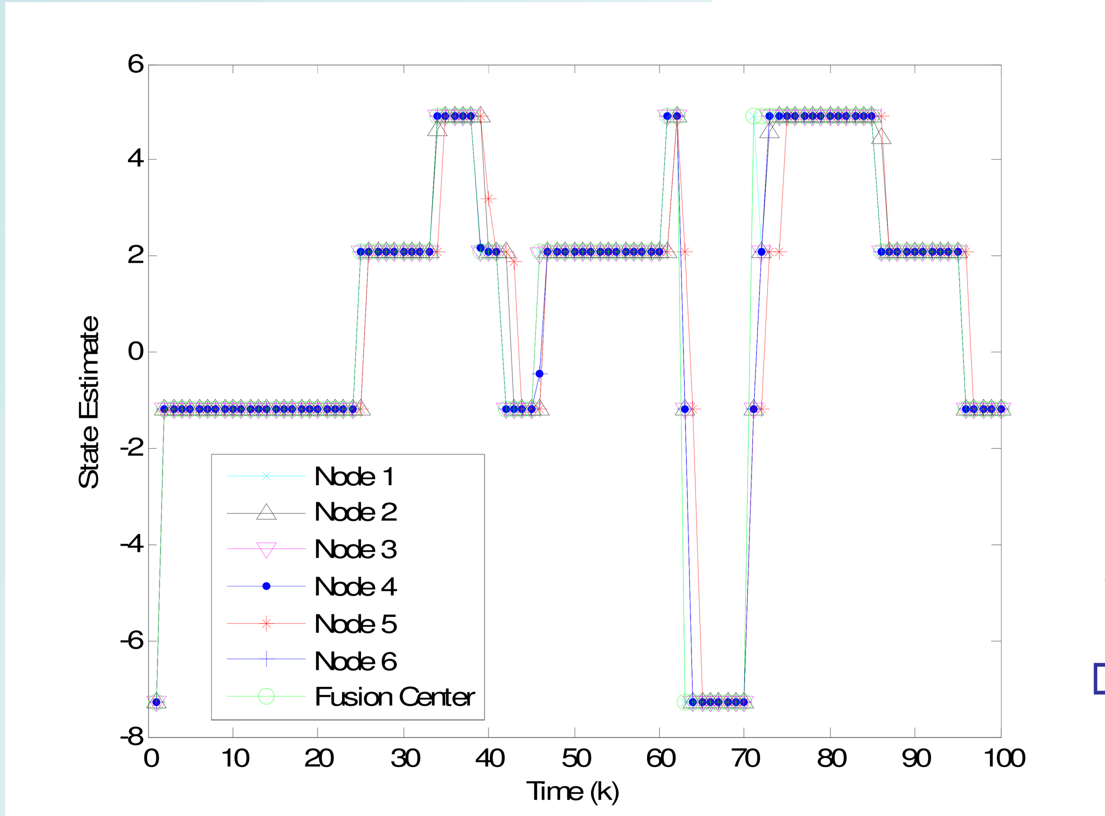
Simulation Results



Network Topology



Convergence in mean of the distributed state estimate to the centralized one



Distributed and centralized state estimates